

# NONEXPANDING ATTRACTORS: CONJUGACY TO ALGEBRAIC MODELS AND CLASSIFICATION IN 3-MANIFOLDS

AARON W. BROWN

**ABSTRACT.** We prove a result motivated by Williams's classification of expanding attractors and the Franks-Newhouse Theorem on codimension-1 Anosov diffeomorphisms: If  $\Lambda$  is a topologically mixing hyperbolic attractor with  $\dim E^u|_{\Lambda} = 1$  then either  $\Lambda$  is expanding or is homeomorphic to a compact abelian group (a *toral solenoid*); in the latter case  $f|_{\Lambda}$  is conjugate to a group automorphism. As a corollary we obtain a classification of all 2-dimensional basic sets in 3-manifolds. Furthermore we classify all topologically mixing hyperbolic attractors in 3-manifolds in terms of the classically studied examples, answering a question of Bonatti in [1].

## 1. INTRODUCTION

In the study of hyperbolic dynamics, a major theme is that strong dynamical hypotheses impose a conjugacy between an abstract dynamical system and an algebraic, or at least highly structured, model. For instance, results of Franks and Manning established that every Anosov diffeomorphism of an infranil-manifold is conjugate to a hyperbolic infranil-automorphism [14, Theorem C]. Among of the oldest conjectures in modern dynamics is the hypothesis that every Anosov diffeomorphism is conjugate to a hyperbolic infranil-automorphism. A partial result towards this conjecture was obtained by Franks and Newhouse for codimension-1 Anosov systems. Recall an Anosov diffeomorphism is called *codimension-1* if  $\dim(E^{\sigma}) = 1$  for some  $\sigma \in \{s, u\}$ .

**Theorem I** (Franks-Newhouse [6], [15]; see also [9]). *Let  $f: M \rightarrow M$  be a codimension-1 Anosov diffeomorphism. Then  $M$  is homeomorphic to a torus, and  $f$  is conjugate to a hyperbolic toral automorphism.*

Outside the realm of global hyperbolicity, that is, when dealing with proper hyperbolic subsets  $\Lambda \subset M$ , one often sees dynamics which is not conjugate to any algebraic system. However, in the case of expanding attractors, Williams showed in [23] that the restricted dynamics  $f|_{\Lambda}$  is conjugate to the shift map on a generalized solenoid. Recall that by an *expanding attractor* we mean a hyperbolic attractor  $\Lambda$  such that  $\dim(\Lambda) = \dim(E^u|_{\Lambda})$ . Also by a *generalized solenoid* (or *n-solenoid*) we mean a topological space  $N$  (which Williams takes to be a *branched n-manifold*), and a surjective map  $g: N \rightarrow N$ , and define the generalized solenoid to be the inverse limit

$$\varprojlim(N, g) := \varprojlim\{N \xleftarrow{g} N \xleftarrow{g} N \xleftarrow{g} \dots\}$$

---

1991 *Mathematics Subject Classification.* Primary: 37C70, 37C15; Secondary: 37D20.  
*Key words and phrases.* Hyperbolic attractors, solenoids, conjugacy, classification.

with the natural shift map  $\sigma$ . (See Section 4 for the construction of the inverse limit in a more specific setting adapted to our problem.)

**Theorem II** ([23, Theorem A]). *Assume  $\Lambda$  is an  $n$ -dimensional expanding attractor for  $f \in \text{Diff}(M)$ . Then  $f|_{\Lambda}$  is conjugate to the shift map of an  $n$ -solenoid.*

Note that Theorem II, as originally stated in [23], required the additional hypothesis that the foliation  $\{W_{\epsilon}^s(x) \mid x \in \Lambda\}$  was  $C^1$  on some neighborhood of  $\Lambda$ . This was later seen to be unnecessary (see for example [2]). While not algebraic, the conjugacy in Theorem II provides a significant insight into the topology of  $\Lambda$  and the dynamics of  $f|_{\Lambda}$ .

In this article we present a result inspired in part by the Franks-Newhouse Theorem on codimension-1 Anosov diffeomorphisms, and somewhat dual to the conjugacy between the dynamics of 1-dimensional expanding attractors and shift maps on generalized solenoids established in [21] and [22]. In particular, we study *non-expanding* hyperbolic attractors  $\Lambda$  for an embedding  $f$ , under the assumption that  $\dim E^u|_{\Lambda} = 1$ , and show that the dynamics  $f|_{\Lambda}$  is conjugate to an automorphism of a compact abelian group. We take our dynamics to be generated by  $C^r$  embeddings for  $r \geq 1$ .

**Theorem 1.1.** *Let  $\Lambda \subset U \subset M$  be a compact topologically mixing hyperbolic attractor for a  $C^r$  embedding  $f: U \rightarrow M$  such that  $\dim E^u|_{\Lambda} = 1$ . Then either  $\Lambda$  is expanding, or is an embedded toral solenoid (see Section 4). In the latter case,  $f|_{\Lambda}$  is conjugate to a leaf-wise hyperbolic solenoidal automorphism. In particular, if  $\Lambda$  is locally connected then  $\Lambda$  is homeomorphic to a torus and  $f|_{\Lambda}$  is conjugate to a hyperbolic toral automorphism.*

Using the primary result in [11] we conclude that the only 2-dimensional toral solenoids that may be embedded in a 3-manifold are homeomorphic to  $\mathbb{T}^2$ . In particular, we obtain the following.

**Corollary 1.2.** *Let  $M$  be a 3-manifold, and let  $\Lambda \subset M$  be a basic set with  $\dim(\Lambda) = 2$ . Then either  $\Lambda$  is a codimension-1 expanding attractor (or contracting repeller), or  $\Lambda$  decomposes as a disjoint union*

$$\Lambda = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_k$$

*where each  $\Omega_j$  is homeomorphic to  $\mathbb{T}^2$  and  $f^k|_{\Omega_j}$  is conjugate to a hyperbolic automorphism of  $\mathbb{T}^2$ .*

We note that the above corollary is a significantly stronger version of the main result in [7]. Indeed, in [7] the result corresponding to the second case in Corollary 1.2 requires the additional hypothesis that  $\Lambda$  is embedded as a subset of a closed surface in  $M$ . Our result, on the other hand, rules out the possibility that  $\dim(E^u|_{\Lambda}) = 1$  and  $W^s(x) \cap \Lambda$  is a connected 1-dimensional set that is not a manifold, for example, a Sierpinski carpet.

It should also be noted that in the conclusion of Corollary 1.2, the  $\mathbb{T}^2$  need not be smoothly embedded. Indeed in [12] a hyperbolic attractor is constructed as a nowhere differentiable torus embedded in a 3-manifold.

The motivation for this work was initially to answer a question by Bonatti [1] which can be paraphrased as follows: Do there exist examples of hyperbolic attractors in 3-manifolds besides the classical examples? We answer this question in the negative.

**Theorem 1.3.** *Let  $M$  be a 3-manifold, and let  $\Lambda \subset U \subset M$  be a topologically mixing, hyperbolic attractor for a  $C^r$  embedding  $f: U \rightarrow M$ . If*

- $\dim \Lambda = 0$ : then  $\Lambda$  is an attracting fixed point for  $f$ ;*
- $\dim \Lambda = 1$ : then  $\dim E^u|_{\Lambda} = 1$  and  $\Lambda$  is conjugate to the shift map on a generalized 1-solenoid as classified by Williams ([22]);*
- $\dim \Lambda = 2$ : then we have  $1 \leq \dim E^u|_{\Lambda} \leq 2$  and if*
  - $\dim E^u|_{\Lambda} = 1$ : then  $\Lambda$  is homeomorphic to  $\mathbb{T}^2$  and  $f|_{\Lambda}$  is conjugate to a hyperbolic toral automorphism;*
  - $\dim E^u|_{\Lambda} = 2$ : then  $\Lambda$  is a codimension-1 expanding attractor studied by Plykin ([17], [18]);*
- $\dim \Lambda = 3$ : then  $\Lambda = M \cong \mathbb{T}^3$  and  $f$  is conjugate to a hyperbolic toral automorphism.*

We remark that in the case of 1-dimensional topologically mixing attractors (which are necessarily expanding), the proof of Theorem 1.1 provides a mechanism to determine if the attractor is algebraic, that is, if  $f|_{\Lambda}$  is conjugate to a solenoidal automorphism. In particular the presence or absence of a *global product structure* as described in Section 5.3.2 determines whether or not a 1-dimensional attractor is algebraic. See Proposition 5.30.

## 2. HYPERBOLIC DYNAMICS

We begin with background material in hyperbolic dynamics and attractors. Let  $M$  be a smooth manifold endowed with a Riemannian metric. Given  $U \subset M$  and a  $C^r$  embedding  $f: U \rightarrow M$ ,  $r \geq 1$ , we say a subset  $\Lambda \subset U$  is *invariant* if  $f(\Lambda) = \Lambda$ . A compact invariant set  $\Lambda$  is said to be *hyperbolic* if there exist a Riemannian metric on  $M$  (called the *adapted* metric), a constant  $\kappa < 1$ , and a continuous  $Df$ -invariant splitting of the tangent bundle  $T_x M = E^s(x) \oplus E^u(x)$  over  $\Lambda$  so that for every  $x \in \Lambda$  and  $n \in \mathbb{N}$

$$\begin{aligned} \|Df_x^n v\| &\leq \kappa^n \|v\|, \quad \text{for } v \in E^s(x) \\ \|Df_x^{-n} v\| &\leq \kappa^n \|v\|, \quad \text{for } v \in E^u(x). \end{aligned}$$

We set

$$V^{\pm} = \bigcap_{n \in \mathbb{N}} f^{\pm n}(U).$$

When  $\Lambda$  is hyperbolic, there exists an  $\epsilon > 0$  such that the sets

$$\begin{aligned} W_{\epsilon}^s(x) &:= \{y \in V^- \mid d(f^n(x), f^n(y)) < \epsilon, \text{ for all } n \geq 0\} \\ W_{\epsilon}^u(x) &:= \{y \in V^+ \mid d(f^{-n}(x), f^{-n}(y)) < \epsilon, \text{ for all } n \geq 0\} \end{aligned}$$

are  $C^r$  embedded open disks, called the *local stable* and *unstable* manifolds. Furthermore, if  $d$  is the distance on  $M$  induced by the adapted metric, there are  $\lambda < 1 < \mu$  so that for  $x \in \Lambda$ ,  $y \in W_{\epsilon}^s(x)$ ,  $z \in W_{\epsilon}^u(x)$  and  $n \geq 0$  we have

$$\begin{aligned} (1) \quad & d(f^n(x), f^n(y)) \leq \lambda^n d(x, y) \\ (2) \quad & d(f^{-n}(x), f^{-n}(z)) \leq \mu^{-n} d(x, z). \end{aligned}$$

Note that (1) and (2) imply  $f(W_{\epsilon}^s(f^{-1}(x))) \subset W_{\epsilon}^s(x)$  and  $W_{\epsilon}^u(x) \subset f(W_{\epsilon}^u(f^{-1}(x)))$ .

For  $x \in \Lambda$  we also have the sets

$$W^s(x) := \{y \in V^- \mid d(f^n(x), f^n(y)) \rightarrow 0 \text{ as } n \rightarrow \infty\}$$

and

$$W^u(x) := \{y \in V^+ \mid d(f^{-n}(x), f^{-n}(y)) \rightarrow 0 \text{ as } n \rightarrow \infty\}$$

called the *global* stable and unstable manifolds. Both  $W^u(x)$  and  $W^s(x)$  are  $C^r$  injectively immersed submanifolds. Note that in the case that  $f$  is invertible (that is, when  $f(U) = U$ ), we have  $W^u(x) \cong \mathbb{R}^{\dim E^u(x)}$  and  $W^s(x) \cong \mathbb{R}^{\dim E^s(x)}$ .

An invariant set  $\Lambda$  is said to be *topologically transitive* under  $f$  if it contains a dense orbit. Alternatively, a compact invariant subset  $\Lambda \subset M$  is topologically transitive if for all pairs of nonempty open sets  $U, V \subset \Lambda$ , there is some  $n$  such that  $f^n(U) \cap V \neq \emptyset$ . An invariant set  $\Lambda$  is called *topologically mixing* if for all pairs of nonempty open sets  $U, V \subset \Lambda$ , there is some  $N$  such that  $f^n(U) \cap V \neq \emptyset$  for all  $n \geq N$ .

A hyperbolic set  $\Lambda$  is a *hyperbolic attractor* if there is some open neighborhood  $\Lambda \subset V$  such that  $\bigcap_{n \in \mathbb{N}} f^n(V) = \Lambda$ . Alternatively, if  $\Lambda$  is a hyperbolic set, then it is an attractor if and only if  $W^u(x) \subset \Lambda$  for all  $x \in \Lambda$ . When  $\Lambda$  is a hyperbolic attractor, the set  $\bigcup_{y \in \Lambda} W^s(y)$  is called the *basin* of  $\Lambda$ . Note that if  $\Lambda$  is a topologically mixing hyperbolic attractor, then for each  $x \in \Lambda$ ,  $W^u(x)$  is dense in  $\Lambda$ .

We recall from the introduction that a hyperbolic attractor  $\Lambda$  is called *expanding* if the topological dimension of  $\Lambda$  equals the dimension of the unstable manifolds. (For an introduction to topological dimension, see [10].) Alternatively,  $\Lambda$  is expanding if for every  $x \in \Lambda$  the set  $W_\epsilon^s(x) \cap \Lambda$  is totally disconnected.

**2.1. Local product structure and Markov partitions.** Recall that given a compact hyperbolic set, we may find  $0 < \delta < \eta$  so that  $d(x, y) < \delta$  implies the intersection  $W_\eta^u(x) \cap W_\eta^s(y)$  is a singleton. We say that a hyperbolic set  $\Lambda$  has *local product structure* if for  $\eta, \delta$  above,  $d(x, y) < \delta$  implies  $W_\eta^u(x) \cap W_\eta^s(y) \subset \Lambda$ . A compact hyperbolic set  $\Lambda$  is called *locally maximal* if there exists an open set  $\Lambda \subset V$  such that  $\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(V)$ . For compact hyperbolic sets, local maximality is equivalent to the existence of a local product structure [13]; in particular, hyperbolic attractors have local product structure.

**Definition 2.1.** Given a set  $\Lambda$  with local product structure and  $\delta$  and  $\eta$  as above, we say a closed set  $R \subset \Lambda$  is a *rectangle* or a *local product chart* if

- (1)  $\sup\{d(x, y) \mid x, y \in R\} < \delta$ ;
- (2)  $R$  is proper, that is,  $R$  is equal to the closure of its interior (in  $\Lambda$ );
- (3)  $x, y \in R$  implies  $W_\eta^u(x) \cap W_\eta^s(y) \subset R$ .

If  $R$  is a rectangle, we write  $W_R^\sigma(x) := W_\eta^\sigma(x) \cap R$ .

If  $\Lambda$  is an attractor, we say an ambiently open set  $V \subset M$  is a *local  $s$ -product neighborhood* if the closure of  $V \cap \Lambda$  is a rectangle and for each  $x \in V \cap \Lambda$

$$V \subset \bigcup_{y \in W_\delta^u(x)} W_\delta^s(y).$$

For  $x \in V \cap \Lambda$  we notate

$$W_V^s(x) := W_\delta^s(x) \cap V.$$

**Definition 2.2.** Given a hyperbolic set  $\Lambda$  with local product structure we say a collection of rectangles  $\mathcal{R} = \{R_j\}$  is a *Markov partition* if

- (1)  $\Lambda = \bigcup_j R_j$ ;
- (2) for  $i \neq j$ ,  $R_j \cap R_i \subset \partial R_j$  where  $\partial$  denotes the topological boundary;

(3)  $x \in R_i \in \mathcal{R}$  and  $f(x) \in \text{int}(R_j)$  implies

$$f(W_{R_i}^s(x)) \subset R_j$$

and  $f^{-1}(x) \in \text{int}(R_j)$  implies

$$f^{-1}(W_{R_i}^u(x)) \subset R_j.$$

We note that every locally maximal hyperbolic set admits a Markov partition; in particular, hyperbolic attractors admit Markov partitions. Also note that if  $R$  is a rectangle and  $\mathcal{R}$  is a Markov partition, then  $f(R)$  is a rectangle and  $f(\mathcal{R}) := \{f(R_j)\}$  is a Markov partition. In particular, we have the following.

**Claim 2.3.** *If  $\Lambda$  is locally maximal, then given any set  $K \subset W^\sigma(x) \cap \Lambda$ , compact in the internal topology of  $W^\sigma(x)$ , there is a rectangle containing  $K$ .*

**2.2. Disintegration of the measure of maximal entropy.** For a hyperbolic set with local product structure we define a *canonical isomorphism* between subsets of the stable and unstable manifolds.

**Definition 2.4** (Canonical Isomorphism). Let  $\Lambda$  be a locally maximal hyperbolic set,  $R$  a rectangle, and  $x \in R$ . Let  $x' \in W_R^u(x)$ , and let  $D \subset W_R^s(x)$ ,  $D' \subset W_R^s(x')$ . Then  $D$  and  $D'$  are said to be *canonically isomorphic* if  $y \in D \cap \Lambda$  implies  $D' \cap W_R^u(y) \neq \emptyset$  and  $y' \in D' \cap \Lambda$  implies  $D \cap W_R^u(y') \neq \emptyset$ .

Similarly, we may define a canonical isomorphism between subsets of local unstable manifolds.

Recall that a point  $x \in M$  is said to be *nonwandering* if for every neighborhood  $U$  of  $x$ , there is an  $n$  so that  $f^n(U) \cap U \neq \emptyset$ . Let  $\text{NW}(f)$  denote the nonwandering points of  $f$ . Recall that given an Axiom-A diffeomorphism, (respectively a locally maximal hyperbolic set  $\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(V)$ ) we have a partition, called the *spectral decomposition*, of the nonwandering points  $\text{NW}(f) = \Omega_1 \cup \dots \cup \Omega_k$  (respectively  $\text{NW}(f|_\Lambda) = \Omega_1 \cup \dots \cup \Omega_k$ ) where each  $\Omega_j$  is a transitive hyperbolic set for  $f$  (see [13], [20]). Given a spectral decomposition, we call the partition elements  $\Omega_j$  above *basic sets*. That is, a compact hyperbolic set  $\Omega \subset \text{NW}(f)$  is a *basic set*, if  $\Omega$  is open in  $\text{NW}(f)$  and  $f$  is topologically transitive on  $\Omega$ . Clearly topologically mixing hyperbolic attractors are basic sets.

Given a basic set, there is a canonical disintegration of the measure of maximal entropy as a product of measures supported on the stable and unstable manifolds. The following is adapted from [19].

**Theorem 2.5** (Ruelle, Sullivan [19]). *Let  $\Omega$  be a basic set for  $f$ . Let  $h$  be the topological entropy of  $f|_\Omega$ . Then there is an  $\epsilon > 0$  so that for each  $x \in \Omega$  there is a measure  $\mu_x^u$  on  $W_\epsilon^u(x)$  and a measure  $\mu_x^s$  on  $W_\epsilon^s(x)$  such that:*

- a)  $\text{supp}(\mu_x^u) = W_\epsilon^u(x) \cap \Omega$  and  $\text{supp}(\mu_x^s) = W_\epsilon^s(x) \cap \Omega$ ,
- b)  $\mu_x^u$  and  $\mu_x^s$  are invariant under canonical isomorphism (see Definition 2.4); that is, if  $x' \in W_\eta^s(x)$  and  $D \subset W_\eta^u(x)$ ,  $D' \subset W_\eta^u(x')$  are canonically isomorphic then  $\mu_x^u(D) = \mu_{x'}^u(D')$ , and if  $x' \in W_\eta^u(x)$  and  $D \subset W_\eta^s(x)$ ,  $D' \subset W_\eta^s(x')$  are canonically isomorphic then  $\mu_x^s(D) = \mu_{x'}^s(D')$ ,
- c)  $f_*\mu_x^u = e^{-h}\mu_{f(x)}^u$  and  $f_*^{-1}\mu_x^s = e^{-h}\mu_{f^{-1}(x)}^s$ ,
- d) the product measure  $\mu_x^u \times \mu_x^s$  is locally equal to Bowen's measure of maximal entropy.

By 2.5(b) we drop the subscript and simply write  $\mu^\sigma$ . By additivity, we may extend the definition of  $\mu^\sigma$  to any set  $K \subset W^\sigma(x)$  for  $\sigma \in \{s, u\}$ . The following properties of  $\mu^\sigma$  are corollaries to the proof of Theorem 2.5 in [19].

**Corollary 2.6.** *Let  $\Omega$  be a basic set with an infinite number of points. Then for  $\sigma \in \{s, u\}$*

- $\mu^\sigma$  is non-atomic and positive on non-empty open sets in  $W^\sigma(x) \cap \Lambda$ ;
- $\mu^\sigma(K)$  is finite for sets  $K \subset W^\sigma(x)$  compact in the internal topology of  $W^\sigma(x)$ .

Furthermore, in the case  $\dim E^u|_\Lambda = 1$ , we have the following.

**Corollary 2.7.** *Let  $\Lambda$  be a hyperbolic attractor such that  $\dim E^u|_\Lambda = 1$ . Then for any connected set  $K \subset W^u(x)$  (that is, an interval) we have  $\mu^u(K) < \infty$  if and only if its closure  $\overline{K}$  in  $W^u(x)$  is compact in the internal topology of  $W^u(x)$ .*

*Proof.* If  $\overline{K}$  is not compact in  $W^u(x)$ , then  $K$  passes through some rectangle a countable number of times which implies that  $\mu^u(K) = \infty$ .  $\square$

### 3. LIMITS OF DIRECTED AND INVERSE SYSTEMS

We review basic constructions and properties of the direct and inverse limit objects in algebra and topology.

**3.1. Direct limits.** Given a topological space  $X$  and an injective continuous map  $f: X \rightarrow X$  we construct the *direct limit*

$$\varinjlim(X, f) := \varinjlim \{X \xrightarrow{f} X \xrightarrow{f} X \xrightarrow{f} \dots\}$$

as follows. Endow  $\mathbb{N}$  with the discrete topology and introduce the equivalence relation on  $X \times \mathbb{N}$  generated by the relation  $(x, k) \sim (f(x), k+1)$ . Then we define

$$\varinjlim(X, f) := (X \times \mathbb{N}) / \sim.$$

The map  $f: X \rightarrow X$  naturally induces a homeomorphism  $\tau_f: \varinjlim(X, f) \rightarrow \varinjlim(X, f)$  by

$$\tau_f: [(x, m)] \mapsto [(f(x), m)].$$

Note, that for  $m \geq 1$  we have  $\tau_f([(x, m)]) = [(x, m-1)]$ , whence it is natural to refer to  $\tau_f$  as the *left shift* on  $\varinjlim(X, f)$ .

We present an alternate, more explicit, construction of the set  $\varinjlim(X, f)$ . For every  $j \in \mathbb{N}$  define a homeomorphism

$$h_j: X_j \rightarrow X$$

and consider the inclusion  $i_j: X_j \hookrightarrow X_{j+1}$  given by  $i_j = h_{j+1}^{-1} \circ f \circ h_j$ . We then have  $X_0 \subset X_1 \subset X_2 \subset \dots$  whence we define

$$(3) \quad \varinjlim \{X \xrightarrow{f} X \xrightarrow{f} \dots\} = \bigcup_{n \in \mathbb{N}} X_n.$$

When the map  $f: X \rightarrow X$  is open, the inclusions  $i_j$  induce a nested inclusion of topologies, the union of which correctly reconstructs the topology of the direct limit. Given  $\xi \in \varinjlim(X, f)$ , we have that  $\xi \in X_j$  for some  $j$  whence we may define

$$\tau_f(\xi) = h_j^{-1} \circ f \circ h_j(\xi).$$

One verifies this definition of  $\tau_f$  coincides with that above.

By the second construction, we see that if  $X$  is a  $C^r$  manifold and  $f$  a  $C^r$  embedding, then  $\varinjlim(X, f)$  can be endowed with a  $C^r$  differential structure under which  $\tau_f$  is a  $C^r$  diffeomorphism.

Given a group  $G$  and a homomorphism  $h: G \rightarrow G$  we define

$$\varinjlim(G, h) := \varinjlim\{G \xrightarrow{h} G \xrightarrow{h} G \xrightarrow{h} \dots\}$$

as follows. Let  $i_j: G_j \rightarrow G$  be a group isomorphism and define  $h_j: G_j \rightarrow G_{j+1}$  by  $h_j: g \mapsto i_{j+1}^{-1}(h(i_j(g)))$ . Let  $N$  be the normal subgroup of  $\bigoplus_{k \in \mathbb{N}} G_k$  generated by the elements  $\{g_j^{-1}h_j(g_j)\}$  for  $g_j \in G_j$ . Then

$$\varinjlim(G, h) = \left( \bigoplus_{k \in \mathbb{N}} G_k \right) / N$$

with canonical left shift automorphism  $\tau_h$  given by  $\tau_h: [g_j] \mapsto [i_j^{-1} \circ h \circ i_j(g_j)]$ . We notate  $[(g, m)] := g_m + N$  for  $g_m \in G_m$ .

The following proposition is straightforward from the Van Kampen theorem.

**Proposition 3.1.** *Let  $X$  be a connected manifold,  $f: X \rightarrow X$  an embedding, and  $G = \pi_1(X)$ . Then*

- $\pi_1(\varinjlim(X, f)) = \varinjlim(G, f_*)$
- $(\tau_f)_*$  is the map given by  $(\tau_f)_*([(g, m)]) = [(f_*(g), m)]$ .

The construction above allows us to embed every hyperbolic attractor as an attractor for an ambient diffeomorphism.

**Claim 3.2.** *Let  $\Lambda \subset U \subset M$  be a hyperbolic attractor for a  $C^r$  embedding  $f: U \rightarrow M$ . Then there is a  $C^r$  diffeomorphism  $f': M' \rightarrow M'$ , a hyperbolic attractor  $\Lambda' \subset M'$  for  $f'$ , neighborhoods  $N$  and  $N'$  of  $\Lambda$  and  $\Lambda'$  respectively, and a  $C^r$  diffeomorphism  $h: N \rightarrow N'$  so that  $h(\Lambda) = \Lambda'$  and  $h \circ f|_N = f'|_{N'} \circ h$ .*

*Proof.* Let  $N = \bigcup_{y \in \Lambda} W_\epsilon^s(y)$ . Then  $f: N \rightarrow N$  is a  $C^r$  embedding. Take  $X = N$  and  $M' := \varinjlim(X, f)$ . Then we have a canonical inclusion  $\Lambda \subset X_0 \cong N$ , where  $X_0$  is as in (3). But then  $\Lambda \subset X_0 \subset M'$  is a hyperbolic attractor for the  $C^r$  diffeomorphism  $\tau_f: M' \rightarrow M'$ .  $\square$

Note that in constructing the direct limit, we assumed the map  $f: X \rightarrow X$  was injective to avoid pathological topological properties in the limiting object.

**3.2. Inverse limits.** Let  $f: X \rightarrow X$  be a continuous map (which we typically take to be surjective). We then define the *inverse limit*

$$\varprojlim(X, f) := \varprojlim\{X \xleftarrow{f} X \xleftarrow{f} X \xleftarrow{f} \dots\}$$

to be the subset of  $X^{\mathbb{N}} := \prod_{i \in \mathbb{N}} X$  satisfying

$$(x_0, x_1, x_2, \dots) \in \varprojlim(X, f) \text{ if } x_j = f(x_{j+1})$$

for all  $j \in \mathbb{N}$ . We then have an induced homeomorphism  $\sigma_f: \varprojlim(X, f) \rightarrow \varprojlim(X, f)$  given by

$$\sigma_f: (x_0, x_1, x_2, \dots) \mapsto (f(x_0), f(x_1), f(x_2), \dots) = (f(x_0), x_0, x_1, x_2, \dots)$$

hence it is natural to call  $\sigma_f$  a *right shift map*. We will call the topological object  $\varprojlim(X, f)$  a *generalized solenoid*.



Note that even in the case that  $X$  is a manifold and  $f$  is a smooth map, we do not expect  $\varprojlim(X, f)$  to have a manifold structure. Indeed in the case that  $f$  is a  $C^\infty$  covering with degree greater than 1, the limit  $\varprojlim(X, f)$  will locally be the product of a Cantor set and a manifold.

If  $G$  is a group, and  $h$  a homomorphism we define

$$\varprojlim(G, h) := \varprojlim\{G \xleftarrow{h} G \xleftarrow{h} G \xleftarrow{h} \dots\}$$

to be the subgroup of  $\prod_{n \in \mathbb{N}} G$  satisfying  $(g_0, g_1, g_2, \dots) \in \varprojlim(G, h)$  if  $g_j = h(g_{j+1})$ , with the induced right shift automorphism

$$\sigma_h: (g_0, g_1, g_2, \dots) \mapsto (h(g_0), g_0, g_1, g_2, \dots).$$

#### 4. TORAL SOLENOIDS

We give a brief introduction to *toral solenoids*, the compact abelian groups obtained as the algebraic models in the conclusion of Theorem 1.1. For more detailed exposition, see, for example, [3]. For an explicit construction of toral solenoids embedded as hyperbolic attractors for differentiable dynamics, see [8].

Let  $A \in \text{Mat}(k, \mathbb{Z})$  have non-zero determinant. Then considering the standard torus  $\mathbb{T}^k := \mathbb{R}^k / \mathbb{Z}^k$  as a compact abelian group, the map  $A: \mathbb{R}^k \rightarrow \mathbb{R}^k$  induces an endomorphism  $A: \mathbb{T}^k \rightarrow \mathbb{T}^k$ . We define a *toral solenoid*  $\mathcal{S}_A$  to be the topological group obtained via the inverse limit

$$\mathcal{S}_A := \varprojlim(\mathbb{T}^k, A) = \varprojlim\{\mathbb{T}^k \xleftarrow{A} \mathbb{T}^k \xleftarrow{A} \dots\}.$$

Note the above inverse limit is taken both as a limit of topological and algebraic objects, and that  $\mathcal{S}_A$  inherits the right shift automorphism  $\sigma_A: \mathcal{S}_A \rightarrow \mathcal{S}_A$

$$\sigma_A: (x_0, x_1, x_2, \dots) \mapsto (Ax_0, x_0, x_1, \dots).$$

$\mathcal{S}_A$  will fail to be a manifold in the case when  $|\det(A)| > 1$ , in which case we will call  $\mathcal{S}_A$  *proper*. Let  $\mathcal{C}_\xi$  denote the path component of  $\xi$  in  $\mathcal{S}_A$ . Then, even in the case  $|\det(A)| > 1$ , we can endow the path components  $\{\mathcal{C}_\xi\}$  with the smooth Euclidean structure pulled back from the projection to the zeroth coordinate  $\mathcal{S}_A \rightarrow \mathbb{T}^k$ . With respect to this Euclidean structure the map  $\sigma_A: \mathcal{C}_\xi \rightarrow \mathcal{C}_{\sigma_A(\xi)}$  is smooth. Furthermore, in the case when  $A$  has no eigenvalues of modulus 1, the map  $\sigma_A: \mathcal{C}_\xi \rightarrow \mathcal{C}_{\sigma_A(\xi)}$  is hyperbolic with respect to the pull-back metric, whence we say  $\sigma_A$  is *leaf-wise hyperbolic*.

**4.1.  $\mathbb{R}^k$  and  $\mathbb{Z}^k$  actions.** We now define an  $\mathbb{R}^k$  action and an induced  $\mathbb{Z}^k$  action on  $\mathcal{S}_A$ . (Compare to the immersions of  $\mathbb{R}^k$  and  $\mathbb{Z}^k$  into  $\mathcal{S}_A$  constructed in [3]).

**Definition 4.1.** We define the  $\mathbb{R}^k$  action  $\theta: \mathbb{R}^k \times \mathcal{S}_A \rightarrow \mathcal{S}_A$  by the rule

$$(4) \quad \theta_v: ([y_0], [y_1], [y_2], \dots) \mapsto ([y_0 + v], [y_1 + A^{-1}v], [y_2 + A^{-2}v], \dots)$$

for  $v \in \mathbb{R}^k$  and  $([y_0], [y_1], [y_2], \dots) \in \mathcal{S}_A$  where  $[y]$  denotes the class of  $y \in \mathbb{R}^k$  in the quotient  $\mathbb{T}^k = \mathbb{R}^k / \mathbb{Z}^k$ .

We then define the  $\mathbb{Z}^k$  action  $\vartheta: \mathbb{Z}^k \times \mathcal{S}_A \rightarrow \mathcal{S}_A$  to be the restriction of  $\theta$  to the subgroup  $\mathbb{Z}^k \subset \mathbb{R}^k$ .

Let  $p_0: \mathcal{S}_A \rightarrow \mathbb{T}^k$  denote the projection in the zeroth coordinate.

**Claim 4.2.** *The action  $\theta$  has the following properties.*

- a) *For each  $\xi \in \mathcal{S}_A$ , the  $\theta$ -orbit of  $\xi$  is dense.*



b) The homomorphism  $\sigma_A$  is  $\theta$ -equivariant; that is,

$$\sigma_A(\theta_v(\xi)) = \theta_{Av}(\sigma_A(\xi))$$

and

$$\sigma_A^{-1}(\theta_v(\xi)) = \theta_{A^{-1}v}(\sigma_A^{-1}(\xi)).$$

c) For all  $\xi \in \mathcal{S}_A$  and  $v \in \mathbb{R}^k$  we have

$$p_0(\theta_v(\xi)) = p_0(\xi) + v.$$

Here  $[x] + v := [x + v]$  is the standard  $\mathbb{R}^k$  action on  $\mathbb{T}^k$ .

d)  $\theta$  commutes with the group operation; that is,

$$\theta_v(\xi + \eta) = \theta_v(\xi) + \eta = \xi + \theta_v(\eta)$$

*Proof.* 4.2(a) is essentially [3, Proposition 2.4]. 4.2(b), 4.2(c), 4.2(d) follow from (4).  $\square$

Define  $\Sigma$  to be the 0-dimensional compact group  $\Sigma := p_0^{-1}([0])$ . Note  $\Sigma$  is either the trivial group  $\{1\}$  in the case that  $\det(A) = \pm 1$ , or homeomorphic to a Cantor set in the case  $|\det(A)| > 1$ . By [3, Corollary 2.3] the map  $p_0: \mathcal{S}_A \rightarrow \mathbb{T}^k$  defines a principle  $\Sigma$ -bundle.

Let  $p: \mathbb{R}^k \rightarrow \mathbb{T}^k$  denote the canonical projection. Given an  $m \in \mathbb{Z}^k$ , we may find some curve  $\gamma: [0, 1] \rightarrow \mathbb{R}^k$  with  $\gamma(0) = 0$  and  $\gamma(1) = m$ . Then  $p(\gamma)$  corresponds to a closed curve at  $[0]$ , hence determines an element of  $\pi_1(\mathbb{T}^k, [0])$ . For  $\eta \in \Sigma$  we have that  $\gamma'(t) := \theta_{\gamma(t)}(\eta)$  is the unique lift of  $p(\gamma(t))$  to  $\mathcal{S}_A$  starting at  $\eta$ . Then we have that  $\gamma'(1) = \vartheta_m(\eta)$ . This motivates the construction in the next section.

**4.2. A covering space for  $(\mathcal{S}_A, \sigma_A)$ .** Define the topological group  $\overline{\mathcal{S}}$  to be the product  $\Sigma \times \mathbb{R}^k$ . The group action  $\vartheta$  of  $\mathbb{Z}^k$  on  $\mathcal{S}_A$  induces a group action  $\vartheta$  of  $\mathbb{Z}^k$  on  $\Sigma$ . We define an embedding  $\alpha$  of  $\mathbb{Z}^k$  as a subgroup of  $\overline{\mathcal{S}}$  by

$$\alpha(n) := (\vartheta_{-n}(e), n)$$

where  $e$  is the identity element of  $\Sigma$ . Then  $\alpha$  naturally defines a  $\mathbb{Z}^k$  action on  $\overline{\mathcal{S}}$  by

$$n \cdot (\xi, v) := (\xi, v) + \alpha(n) = (\vartheta_{-n}(\xi), v + n)$$

for  $n \in \mathbb{Z}^k$ ,  $\xi \in \Sigma$ , and  $v \in \mathbb{R}^k$ .

We also define maps  $\overline{\sigma}: \overline{\mathcal{S}} \rightarrow \overline{\mathcal{S}}$  given by

$$\overline{\sigma}: (\xi, v) \mapsto (\sigma_A(\xi), A(v))$$

and  $\overline{q}: \overline{\mathcal{S}} \rightarrow \mathcal{S}_A$  given by

$$\overline{q}: (\xi, v) \mapsto \theta_v(\xi).$$

We check that  $\overline{\sigma}$  is an injective endomorphism. Furthermore,  $\overline{q}$  is seen to be a homomorphism by Claim 4.2(d).

We have the following properties of the above construction.

**Claim 4.3.**  $\overline{\mathcal{S}}, \overline{\sigma}, \overline{q}$ , and  $\alpha$  satisfy

- a)  $N := \alpha(\mathbb{Z}^k)$  is a discrete subgroup isomorphic to  $\mathbb{Z}^k$ ;
- b)  $\ker(\overline{q}) = N$ , whence we have the canonical identification  $\overline{\mathcal{S}}/N \cong \mathcal{S}_A$  as topological groups;
- c)  $\overline{q} \circ \overline{\sigma} = \sigma_A \circ \overline{q}$ ;
- d)  $\overline{\sigma}(x + \alpha(n)) = \overline{\sigma}(x) + \alpha(A(n))$ .

*Proof.* **4.3(a)** is clear. For **4.3(b)**, let  $\bar{q}(\xi, v) = e$  where  $e$  is the identity in  $\Sigma \subset \mathcal{S}_A$ . Then we have

$$\theta_v(\xi) = e.$$

In particular,  $v \in \mathbb{Z}^k$ . Furthermore

$$\alpha(v) = (\vartheta_{-v}(e), v) = (\theta_v^{-1}(e), v) = (\xi, v)$$

hence  $\bar{q}(\xi, v) = e$  implies  $(\xi, v) \in N$ . Similarly for any  $n \in \mathbb{Z}^k$  we have  $\bar{q}(\alpha(n)) = \vartheta_n(\vartheta_{-n}(e)) = e$  hence **4.3(b)** holds.

We have

$$\bar{q} \circ \bar{\sigma}(\xi, v) = \theta_{Av}(\sigma_A \xi) = \sigma_A(\theta_v(\xi)) = \sigma_A \circ \bar{q}(\xi, v)$$

whence **4.3(c)** follows. Finally we have

$$\bar{\sigma}(\alpha(n)) = (\sigma_A(\vartheta_{-n}(e)), A(n)) = (\vartheta_{-A(n)}(\sigma_A(e)), A(n)) = \alpha(A(n))$$

from which **4.3(d)** follows.  $\square$

Now  $\bar{\sigma}: \bar{\mathcal{S}} \rightarrow \bar{\mathcal{S}}$  is an injective homomorphism, but will fail to be surjective whenever  $|\det(A)| > 1$ . We define the topological group  $\tilde{\mathcal{S}}$  as the direct limit

$$\tilde{\mathcal{S}} := \varinjlim \left\{ \bar{\mathcal{S}} \xrightarrow{\bar{\sigma}} \bar{\mathcal{S}} \xrightarrow{\bar{\sigma}} \bar{\mathcal{S}} \xrightarrow{\bar{\sigma}} \dots \right\}$$

and define  $\tilde{\sigma}: \tilde{\mathcal{S}} \rightarrow \tilde{\mathcal{S}}$  to be the left shift automorphism ( $\tau_{\bar{\sigma}}$  in the notation of Section **3.1**) induced by  $\bar{\sigma}: \bar{\mathcal{S}} \rightarrow \bar{\mathcal{S}}$ ; that is,

$$\tilde{\sigma}([(s, l)]) = [(\bar{\sigma}(s), l)] = [(s, l-1)]$$

where the second equality holds for  $l \geq 1$ . Furthermore, we define the torsion-free abelian group

$$\mathbb{Z}^k[A^{-1}] := \varinjlim \left\{ \mathbb{Z}^k \xrightarrow{A} \mathbb{Z}^k \xrightarrow{A} \mathbb{Z}^k \xrightarrow{A} \dots \right\}$$

and the (left shift) group homomorphism  $\tau_A: \mathbb{Z}^k[A^{-1}] \rightarrow \mathbb{Z}^k[A^{-1}]$ . Note that by Claim **4.3(d)** the diagram

$$\begin{array}{ccccccc} \mathbb{Z}^k & \xrightarrow{A} & \mathbb{Z}^k & \xrightarrow{A} & \mathbb{Z}^k & \xrightarrow{A} & \mathbb{Z}^k \xrightarrow{A} \dots \\ \downarrow \alpha & & \downarrow \alpha & & \downarrow \alpha & & \downarrow \alpha \\ \bar{\mathcal{S}} & \xrightarrow{\bar{\sigma}} & \bar{\mathcal{S}} & \xrightarrow{\bar{\sigma}} & \bar{\mathcal{S}} & \xrightarrow{\bar{\sigma}} & \bar{\mathcal{S}} \xrightarrow{\bar{\sigma}} \dots \end{array}$$

commutes whence we may extend the embedding  $\alpha: \mathbb{Z}^k \rightarrow \bar{\mathcal{S}}$  to an embedding  $\tilde{\alpha}$  of  $\mathbb{Z}^k[A^{-1}]$  as a subgroup of  $\tilde{\mathcal{S}}$ . Since  $\alpha(\mathbb{Z}^k)$  is a discrete subgroup of  $\bar{\mathcal{S}}$ , the homomorphism  $\tilde{\alpha}$  embeds  $\mathbb{Z}^k[A^{-1}]$  as a discrete subgroup of  $\tilde{\mathcal{S}}$ . More explicitly we have

$$\tilde{\alpha}([(n, m)]) := [(\alpha(n), m)]$$

which is seen to be well defined by Claim **4.2(b)**. As above, the embedding  $\tilde{\alpha}$  of  $\mathbb{Z}^k[A^{-1}]$  as a subgroup of  $\tilde{\mathcal{S}}$  defines a natural  $\mathbb{Z}^k[A^{-1}]$  action on  $\tilde{\mathcal{S}}$ . We also define a group homomorphism  $\tilde{q}: \tilde{\mathcal{S}} \rightarrow \mathcal{S}_A$  by

$$\tilde{q}: [((\xi, v), l)] \mapsto \sigma_A^{-l}(\bar{q}(\xi, v)).$$

We enumerate properties of the above constructions.

**Proposition 4.4.** *For  $\tilde{\mathcal{S}}, \tilde{\sigma}, \tilde{\alpha}$ , and  $\tilde{q}$  we have*

- a)  $\tilde{N} := \alpha(\mathbb{Z}^k[A^{-1}])$  is a discrete subgroup isomorphic to  $\mathbb{Z}^k[A^{-1}]$ ;

- b)  $\ker(\tilde{q}) = \tilde{N}$ , whence we have the canonical identification  $\tilde{\mathcal{S}}/\tilde{N} \cong \mathcal{S}_A$  as topological groups;
- c)  $\tilde{q} \circ \tilde{\sigma} = \sigma_A \circ \tilde{q}$ ;
- d)  $\tilde{\sigma}(x + \tilde{\alpha}(g)) = \tilde{\sigma}(x) + \tilde{\alpha}(\tau_A(g))$  for  $g \in \mathbb{Z}^k[A^{-1}]$ .

*Proof.* 4.4(a) is clear. To see 4.4(b), let  $\tilde{q}([( (\xi, v), l)]) = e$ . Then we have

$$\sigma_A^{-l}(\theta_v(\xi)) = e.$$

Applying  $\sigma_A^l$  of both sides we have  $\theta_v(\xi) = \sigma_A^l(e) = e$  whence  $v \in \mathbb{Z}^k$ . Taking  $g = [(v, l)]$  we have

$$\tilde{\alpha}(g) = [(\alpha(v), l)] = [((\vartheta_{-v}(e), v), l)] = [((\xi, v), l)]$$

hence  $[((\xi, v), l)] = \tilde{\alpha}(g)$ . Similarly one verifies that  $\tilde{q}(\tilde{\alpha}(g)) = e$  for any  $g \in \mathbb{Z}^k[A^{-1}]$ . Hence 4.4(b) holds.

To see 4.4(c), note that for any  $\bar{s} \in \bar{\mathcal{S}}$  and  $l \in \mathbb{N}$  we have

$$\tilde{q} \circ \tilde{\sigma}([( \bar{s}, l)]) = \sigma_A^{-l}(\tilde{q}(\bar{\sigma}(\bar{s}))) = \sigma_A^{-l}(\sigma_A(\tilde{q}(\bar{s}))) = \sigma_A(\sigma_A^{-l}(\tilde{q}(\bar{s}))) = \sigma_A(\tilde{q}([( \bar{s}, l)])).$$

Finally for  $g = [(n, m)]$  we see

$$\tilde{\sigma}(\tilde{\alpha}(g)) = [(\bar{\sigma}(\alpha(n)), m)] = [(\alpha(A(n)), m)] = \tilde{\alpha}(\tau_A(g))$$

establishing 4.4(d).  $\square$

We thus have that  $\tilde{q}: \tilde{\mathcal{S}} \rightarrow \mathcal{S}_A$  is a covering map and that  $\tilde{\sigma}$  lifts  $\sigma_A$ .

**4.3. Metrization of  $\mathcal{S}_A$ .** We conclude this section with the construction of a canonical metric on  $\mathcal{S}_A$  with respect to which  $\mathcal{S}_A$  behaves (metrically) like a hyperbolic set for  $\sigma_A$  when  $A$  is hyperbolic.

Firstly, let  $\rho$  denote the standard metric on  $\mathbb{R}^k$ . Given a curve  $\gamma: [0, 1] \rightarrow \mathcal{S}_A$ , there is a unique curve  $\gamma': [0, 1] \rightarrow \mathbb{R}^k$  with  $\gamma'(0) = 0$  such that  $\gamma(t) = \theta_{\gamma'(t)}(\gamma(0))$ . If  $x, y \in \mathcal{S}_A$  lie in the same path component, define  $\Gamma(x, y)$  to be the set of all curves  $\gamma: [0, 1] \rightarrow \mathcal{S}_A$  with  $\gamma(0) = x$  and  $\gamma(1) = y$ . Then define

$$\rho(x, y) := \inf_{\gamma \in \Gamma(x, y)} \rho(\gamma'(0), \gamma'(1)).$$

Secondly, for any  $x, y \in \mathcal{S}_A$  we define

$$\mathcal{J}(x, y) := \{j \in \mathbb{Z} \mid p_0(\sigma_A^j(x)) \neq p_0(\sigma_A^j(y))\}$$

and

$$d_\Sigma(x, y) := \sum_{j \in \mathcal{J}(x, y)} 2^j.$$

Note that for any  $x, y \in \mathcal{S}_A$  and  $v \in \mathbb{R}^k$  we have  $d_\Sigma(\theta_v(x), \theta_v(y)) = d_\Sigma(x, y)$  and  $d_\Sigma(\sigma_A(x), \sigma_A(y)) = \frac{1}{2}d_\Sigma(x, y)$ .

Given  $x, y \in \mathcal{S}_A$ , let  $\Xi(x, y)$  be the set of all sequences  $\xi = (x_0, y_0, x_1, y_1, \dots, x_l, y_l)$  such that there is a curve  $\gamma_j \subset \mathcal{S}_A$  with endpoints  $x_j$  and  $y_j$  for  $0 \leq j \leq l$ . Then define

$$l(\xi) := \sum_{0 \leq j \leq l} \rho(x_j, y_j) + \sum_{0 \leq j \leq l-1} d_\Sigma(y_j, x_{j+1})$$

and

$$d(x, y) := \inf_{\xi \in \Xi(x, y)} \{l(\xi)\}.$$

Now, denoting

$$\begin{aligned}\tilde{\Sigma} &:= \varinjlim \{ \Sigma \xrightarrow{\sigma_A} \Sigma \xrightarrow{\sigma_A} \Sigma \xrightarrow{\sigma_A} \dots \} \\ &= \{ (x_0, x_1, \dots) \in \mathcal{S}_A \mid A^j(x_0) = [0] \text{ for some } j \in \mathbb{N} \}\end{aligned}$$

we have  $\tilde{\mathcal{S}} \cong \tilde{\Sigma} \times \mathbb{R}^k$ . Hence if  $A \in \text{Mat}(k, \mathbb{Z})$  is a hyperbolic matrix,  $E^+$  and  $E^-$  denote the expanding and contracting subspaces, then  $\tilde{\mathcal{S}} \cong \tilde{\Sigma} \times E^+ \times E^-$ . Furthermore, if  $\tilde{d}$  is the lift of the metric  $d$  to  $\tilde{\mathcal{S}}$  then  $\tilde{d}$  is equivalent to the product metric  $d_\Sigma|_{\tilde{\Sigma}} \times \rho|_{E^+} \times \rho|_{E^-}$ , hence with respect to  $\tilde{d}$

$$\tilde{\sigma}: E^+ \rightarrow E^+$$

is expanding and

$$\tilde{\sigma}: \tilde{\Sigma} \times E^- \rightarrow \tilde{\Sigma} \times E^-$$

is contracting.

## 5. PROOF OF THEOREM 1.1

Since we are only concerned with the topology of  $\Lambda$  and the dynamics  $f|_\Lambda$  in Theorem 1.1, by Claim 3.2 we assume without loss of generality that  $f: M \rightarrow M$  is a diffeomorphism. To prove Theorem 1.1 we first present some preliminary observations and constructions that will enable us to build the essential dichotomy.

**5.1. Preliminaries.** Let  $\Lambda$  be a compact, topologically mixing, hyperbolic attractor for a diffeomorphism  $f: M \rightarrow M$  such that  $\dim E^u|_\Lambda = 1$ . Let  $B$  denote the basin of  $\Lambda$ ,  $\tilde{B}$  the universal cover of  $B$ ,  $\pi: \tilde{B} \rightarrow B$  the covering projection, and  $\tilde{\Lambda} := \pi^{-1}(\Lambda)$ . Let  $G := \pi_1(B)$  denote the fundamental group of  $B$ , which we identify with the group of deck transformations for the covers  $\pi: \tilde{B} \rightarrow B$  and  $\pi|_{\tilde{\Lambda}}: \tilde{\Lambda} \rightarrow \Lambda$ . Given subsets  $H \subset G$  and  $X \subset B$  denote by  $\mathcal{O}_H(X) := \bigcup_{g \in H} g(X)$  the orbit of  $X$  under  $H$ .

Let  $g$  be a Riemannian metric on  $M$ . Note that for any lift  $\tilde{f}$  of  $f$ ,  $\tilde{\Lambda}$  is a hyperbolic set under the pull-back metric  $\pi^*(g)$ . For  $x \in \tilde{\Lambda}$  and  $\sigma \in \{s, u\}$  denote by  $W^\sigma(x)$  the  $\sigma$ -manifold of  $x$  under the dynamics  $\tilde{f}$  in the metric  $\pi^*(g)$ . Note also that  $W^\sigma(x)$  is the connected component of  $\pi^{-1}(W^\sigma(\pi(x)))$  containing  $x$ . In addition note that for  $g \in G$  different from the identity,  $W^\sigma(g(x)) \cap W^\sigma(x) = \emptyset$ . Given a subset  $X \subset \tilde{\Lambda}$  we will write  $W^\sigma(X) := \bigcup_{x \in X} W^\sigma(x)$ .

**Definition 5.1.** Let  $x \in \tilde{B}$ . Define  $d_x^s$  to be the distance function on  $W^s(x)$  induced by restricting the metric  $\pi^*(g)$  to  $W^s(x)$ . For simplicity we shall suppress the dependence on  $x$  and simply write  $d^s(x, y) := d_x^s(x, y)$  whenever  $y \in W^s(x)$ .

Note that  $\tilde{B}$  admits a codimension-1 foliation  $\mathcal{W}^s$  by the stable manifolds of  $\tilde{\Lambda}$ . Since  $\pi_1(\tilde{B}) = \{1\}$ , the foliation  $\mathcal{W}^s$  is transversely orientable. (Indeed, one can always make  $\mathcal{W}^s$  and  $\tilde{B}$  orientable by passing to double covers.) Fix a transverse orientation for  $\mathcal{W}^s$ . Note that neither  $G$  nor any lift of  $f$  is assumed to preserve this transverse orientation.

Given a compact, oriented,  $C^1$  curve  $\gamma \subset \tilde{B}$  that is everywhere transverse to the foliation  $\mathcal{W}^s$ , we define a signed length  $l^u(\gamma)$ . Let  $\{V_i\}$  be a cover of  $\Lambda$  by local  $s$ -product neighborhoods (see Definition 2.1) and let  $\{\tilde{V}_{ij}\} = \pi^{-1}(\{V_i\})$  where for each  $j$ , the set  $\tilde{V}_{ij}$  is homeomorphic to  $V_i$ . Let  $\tilde{f}$  be a lift of  $f$ . Then we may find some  $n > 0$  so that  $\gamma \subset \tilde{f}^{-n}(\bigcup_{ij} \tilde{V}_{ij})$ .

**Definition 5.2.** Given  $\gamma$  and  $n$  as above, we define the *signed unstable length*  $l^u(\gamma)$  as follows. We first define  $l^u$  on a connected component  $\gamma'$  of  $(\gamma \cap \tilde{f}^{-n}(\tilde{V}_{ij}))$ . Let

$$C = \pi(\tilde{f}^n(\gamma'))$$

and define

$$l^u(\gamma') = \text{sgn}(\gamma) e^{-nh} \mu^u(\{y \in W_{V_i}^u(z) \mid W_{V_i}^s(y) \cap C \neq \emptyset\})$$

where  $z$  is any point in  $V_i \cap \Lambda$ ,  $\mu^u$  is as in Theorem 2.5,

$$\text{sgn}(\gamma) = \begin{cases} 1, & \text{if } \gamma \text{ is positively orientated with respect to } \mathcal{W}^s, \\ -1, & \text{if } \gamma \text{ is negatively orientated with respect to } \mathcal{W}^s, \end{cases}$$

and  $h$  is the topological entropy of  $f|_\Lambda$ . We may then extend  $l^u$  to all of  $\gamma$  additively.

Theorem 2.5 shows that  $l^u$  is well defined, independent of all choices above. Furthermore, given any piecewise-smooth oriented curve  $\gamma \subset \tilde{B}$  we may partition  $\gamma$  into a family of curves  $\{\gamma_i\}$ , each of which is everywhere tangent to  $\mathcal{W}^s$  or everywhere transverse to  $\mathcal{W}^s$ ; in the former case we define  $l^u(\gamma_i) = 0$  while in the latter we use Definition 5.2. Thus we may extend the definition of  $l^u$  to all piecewise-smooth oriented curve in  $\tilde{B}$ . Note that by Corollary 2.6,  $l^u(\gamma)$  is non-zero on any curve  $\gamma$  transverse to  $\mathcal{W}^s$  and  $l^u(\{x\}) = 0$ .

Now piecewise-smooth curves generate the group of piecewise-smooth simplicial 1-chains. Thus we may extend the function  $l^u$  to a piecewise-smooth simplicial 1-cochain denoted by  $\alpha^u$ .

**Claim 5.3.** *The cochain  $\alpha^u$  is closed, hence exact.*

*Proof.* A 1-cochain is closed if it is locally independent of path. This is clear by Theorem 2.5(b).  $\square$

Claim 5.3 has the following two corollaries.

**Corollary 5.4.** *Given  $x, y \in \tilde{B}$  and two oriented piecewise-smooth curves  $\gamma_1, \gamma_2$  with end points  $x$  and  $y$  then  $|l^u(\gamma_1)| = |l^u(\gamma_2)|$ .*

*Proof.* Changing orientation if necessary we may assume that the concatenation  $\gamma_1 \cdot \gamma_2$  is a closed 1-chain. But then we have

$$0 = \alpha^u(\gamma_1 \cdot \gamma_2) = l^u(\gamma_1) + l^u(\gamma_2)$$

hence  $l^u(\gamma_2) = -l^u(\gamma_1)$ .  $\square$

**Corollary 5.5.** *For each pair  $x, y \in \tilde{\Lambda}$ , the intersection  $W^s(x) \cap W^u(y)$  contains at most one point.*

*Proof.* If not we could find a piecewise smooth 1-cycle  $\gamma$  with  $|\alpha^u(\gamma)| > 0$ , a contradiction since  $\alpha^u$  is exact.  $\square$

The above corollaries motivate the following definitions.

**Definition 5.6.** We say a subset  $V \subset \tilde{\Lambda}$  is a *product chart* if  $x, y \in V$  implies  $W^u(x) \cap W^s(y)$  is non-empty and  $W^u(x) \cap W^s(y) \subset V$ .

**Definition 5.7.** For  $x \in \tilde{\Lambda}$  and  $x' \in W^u(x)$  let  $l^u(x, x') := l^u(\gamma_{xx'})$  where  $\gamma_{xx'}$  is the unique oriented curve in  $W^u(x)$  from  $x$  to  $x'$ . For  $x \in \tilde{\Lambda}$  and  $L \in \mathbb{R}$ , let  $x +_u L$  denote the unique point  $x' \in W^u(x)$  with  $l^u(x, x') = L$ .

**Definition 5.8.** Given  $x, y \in \tilde{B}$  we define the pseudometric  $d^u(x, y) := |l^u(\gamma)|$  for any piecewise smooth curve  $\gamma$  with endpoints  $x$  and  $y$ . Furthermore we define a metric on leaves of the foliation  $\mathcal{W}^s$  by  $d^u(W^s(x), W^s(y)) := d^u(x, y)$ .

Note that Corollary 5.4 guarantees  $d^u$  is well defined, and Corollaries 2.6 and 2.7 guarantees that the restriction of  $d^u$  to  $W^u(x)$  defines a complete metric consistent with the topology on  $W^u(x)$ .

**Definition 5.9.** For  $x, y \in \tilde{\Lambda}$  we define  $\Xi(x, y)$  to be the set of sequences  $\xi = (x = x_0, y_0, \dots, x_k, y_k = y)$  where  $y_j \in W^u(x_j)$  for  $0 \leq j \leq k$  and  $x_{j+1} \in W^s(y_j)$  for  $0 \leq j \leq k-1$ . Then define

$$d(x, y) := \inf_{\xi \in \Xi(x, y)} \left\{ \sum_{j=0}^k d^u(x_j, y_j) + \sum_{j=0}^{k-1} d^s(x_{j+1}, y_j) \right\}$$

where  $d^s$  is the distance induced by the Riemannian metric in Definition 5.1 and  $d^u$  is the pseudometric constructed from the measure  $\mu^u$  in Definition 5.8. Clearly  $d$  defines a metric on  $\tilde{\Lambda}$  consistent with the ambient topology.

5.1.1. *Global product relation.* We now define a binary relation on points in  $\tilde{\Lambda}$ .

**Definition 5.10** (Global Product Relation). For  $x, y \in \tilde{\Lambda}$  we say  $x \sim y$  if  $y \in W^s(x)$  and

$$W^u(y) \cap W^s(x') \neq \emptyset$$

for all  $x' \in W^u(x)$ .

**Claim 5.11.**  $\sim$  is an equivalence relation.

*Proof.* Clearly  $\sim$  is reflexive. To see that  $\sim$  is symmetric suppose  $x \sim y$ , and that there exists some  $y' \in W^u(y)$  such that  $W^s(y') \cap W^u(x) = \emptyset$ . Set  $L = l^u(y, y')$  and  $x' = x +_u L$ . Then

$$l^u(y, W^u(y) \cap W^s(x')) = L$$

hence  $y' = W^u(y) \cap W^s(x')$  contradicting the assumptions on  $y'$ . Thus  $\sim$  is symmetric. A similar argument shows that  $\sim$  is transitive.  $\square$

We let  $[x]$  denote the equivalence class of  $x$  under the relation  $\sim$ .

**Remark.** The equivalence class  $[x]$  represents the maximal subset of  $\tilde{\Lambda} \cap W^s(x)$  with global product structure, that is, admitting the canonical homeomorphism

$$[x] \times W^u(x) \cong W^u([x])$$

given by

$$(y, x') \mapsto W^u(y) \cap W^s(x').$$

Furthermore we have that  $W^u$  saturation of  $[x]$  is  $\sim$ -saturated whence we have the equality

$$W^u([x]) = [W^u(x)]$$

and  $W^u([x])$ , with the quotient topology, is homeomorphic to  $\mathbb{R}$ .

We enumerate a number of properties of the equivalence classes of  $\sim$ .

**Claim 5.12.**

- a) The equivalence classes are preserved under  $u$ -holonomy; in particular, the  $\mathbb{R}$ -action  $x \mapsto x +_u L$  on  $\tilde{\Lambda}$  descends to a well defined  $\mathbb{R}$  action  $[x] \mapsto [x +_u L] = [x] +_u L$ .
- b) Equivalence classes are preserved by the covering action of  $G$  and by any lift  $\tilde{f}$  of  $f$ .
- c) The equivalence classes of  $\sim$  are closed, both as subsets of the stable manifolds and hence as subsets of  $\tilde{\Lambda}$ .
- d) Let  $C^s(x)$  denote the connected component of  $\tilde{\Lambda} \cap W^s(x)$  containing  $x$ . Then  $C^s(x) \subset [x]$ .
- e) Let  $y \in W^s(x)$  be such that  $C^s(y)$  contains points arbitrarily close to  $C^s(x)$ . Then  $C^s(y) \subset [x]$ .

*Proof.* 5.12(a) and 5.12(b) are trivial.

To see 5.12(c) let  $x_j \rightarrow x$  in  $W^s(x)$  where  $x_j \sim x_k$  for all  $j, k \in \mathbb{N}$ . Suppose there is some  $x' \in W^u(x)$  so that  $W^s(x') \cap W^u(x_j) = \emptyset$  for some (hence all)  $j$ . Let  $C \subset W^u(x)$  be a compact connected set containing  $x$  and  $x'$ . Then there is some rectangle  $V \subset \Lambda$  containing  $\pi(C)$  by Claim 2.3. But then there is a product chart  $\tilde{V} \subset \pi^{-1}(V)$  containing  $C$  and  $x_j$  for a sufficiently large  $j$  contradicting the assumption that  $W^s(x') \cap W^u(x_j) = \emptyset$  for all  $j$ . Hence 5.12(c) holds.

Fix an  $L > 0$ . Clearly the set

$$V = \{y \in W^s(x) \mid W^s(y +_u r) \cap W^u(x) \neq \emptyset \text{ for all } |r| \leq L\}$$

is open in  $W^s(x)$ . By a similar argument as above we see that  $V$  is closed hence  $C^s(x) \subset V$ . Since  $L$  was arbitrary 5.12(d) follows.

For 5.12(e), let  $y$  satisfy the hypotheses and suppose  $y' \in W^u(y)$  is such that  $W^s(y') \cap W^u(x) = \emptyset$  and let  $L = l^u(y, y')$ . Since  $\Lambda$  is compact we may find some  $\delta > 0$  so that for every  $z \in \Lambda$  there is some rectangle  $V(z, L)$  containing both the sets  $W_\delta^s(z) \cap \Lambda$  and  $\pi(\{\tilde{z} +_u r \mid r \leq |L|\})$  where  $\tilde{z}$  is some lift of  $z$  to  $\tilde{\Lambda}$ . By assumption we may find a  $w \in C^s(y)$  and  $x' \in C^s(x)$  so that  $d^s(w, x') < \delta$ ; setting  $w' = w +_u L$  we may find a product chart containing  $w, w'$ , and  $x'$ , hence  $W^s(w') \cap W^u(x') \neq \emptyset$ . By 5.12(a)  $w' \in [y']$  and by 5.12(d)  $x' \in [x]$  whence

$$W^s(y') \cap W^u(x) = W^s(w') \cap W^u(x) = W^s(x' +_u L) \cap W^u(x) \neq \emptyset$$

a contradiction.  $\square$

We now define a metric on the quotient  $\tilde{\Lambda}/\sim$  and study its induced topology.

5.1.2. *Metrization of  $\tilde{\Lambda}/\sim$ .* Denote by  $\tilde{\Omega}$  the set of equivalence classes of  $\tilde{\Lambda}$  under the equivalence relation  $\sim$  and by  $\Omega^s([x])$  the set of equivalence classes of  $\tilde{\Lambda} \cap W^s(x)$  under  $\sim$ . We introduce metric topologies on  $\tilde{\Omega}$  and  $\tilde{\Omega}^s$ . Note that the pseudometric  $d^u$  on  $\tilde{\Lambda}$  descends to a pseudometric on  $\tilde{\Omega}$ ; that is, given two points  $[x], [y] \in \tilde{\Omega}$

$$d^u([x], [y]) := d^u(x, y)$$

is well defined. We define a metric on each  $\Omega^s([x])$  as follows.

**Definition 5.13.** Given  $[x] \in \tilde{\Omega}$  and  $[y] \in \Omega^s([x])$  let

$$r^s([x], [y]) := \sup\{r > 0 \mid W^u(y) \cap W^s(x \pm_u r') \neq \emptyset \quad \forall 0 < r' < r\}$$



and

$$d_{\Omega}^s([x], [y]) = \begin{cases} \frac{1}{r^s([x], [y])}, & r^s([x], [y]) \neq \infty, \\ 0, & r^s([x], [y]) = \infty. \end{cases}$$

Note that  $r^s([x], [y]) = r^s([y], [x])$ ,  $r^s([x], [y]) > 0$ , and that  $r^s([x], [y]) \neq \infty$  unless  $[x] = [y]$ . Furthermore,

**Lemma 5.14.**  $d_{\Omega}^s([x], [y])$  is a metric on  $\Omega^s([x])$ .

*Proof.* We clearly have  $d_{\Omega}^s([x], [y]) = 0$  if and only if  $[x] = [y]$  and  $d_{\Omega}^s([x], [y]) = d_{\Omega}^s([y], [x])$ . Thus we need only prove the triangle inequality

$$(5) \quad d_{\Omega}^s([x], [y]) \leq d_{\Omega}^s([x], [z]) + d_{\Omega}^s([z], [y]).$$

By definition of  $r^s$  we have for  $[y], [z] \in \Omega^s([x])$

$$r^s([x], [y]) \geq \min\{r^s([x], [z]), r^s([z], [y])\}.$$

Thus

$$(6) \quad d_{\Omega}^s([x], [y]) \leq \max\{d_{\Omega}^s([x], [z]), d_{\Omega}^s([z], [y])\}$$

and (5) holds.  $\square$

**Definition 5.15.** Given two points  $[x], [y] \in \tilde{\Omega}$  let  $\Xi([x], [y])$  be the space of all sequences  $([x_0], [y_0], [x_1], [y_1], \dots, [x_k], [y_k])$  in  $\tilde{\Omega}$  with

- (1)  $[x_0] = [x]$ , and  $[y_k] = [y]$
- (2)  $y_j \in W^u(x_j)$  for  $0 \leq j \leq k$
- (3)  $x_j \in W^s(y_{j-1})$  for  $1 \leq j \leq k$ .

Given a  $\xi = ([x_0], [y_0], \dots, [y_k]) \in \Xi([x], [y])$  define

$$l(\xi) := \sum_{j=0}^k d^u(x_j, y_j) + \sum_{j=1}^k d_{\Omega}^s([x_j], [y_{j-1}])$$

and define  $d_{\Omega}([x], [y])$  by

$$d_{\Omega}([x], [y]) := \inf \{l(\xi) \mid \xi \in \Xi([x], [y])\}.$$

Clearly  $d_{\Omega}$  defines a metric on  $\Omega$ .

**Corollary 5.16.** The group  $G = \pi_1(B)$  acts via isometries on  $(\tilde{\Omega}, d_{\Omega})$ . Furthermore the dynamics on  $\tilde{\Omega}$  induced by the dynamics  $\tilde{f}: \Lambda \rightarrow \Lambda$ , which we also denote by  $\tilde{f}: \tilde{\Omega} \rightarrow \tilde{\Omega}$ , acts conformally:  $d_{\Omega}(\tilde{f}([x]), \tilde{f}([y])) = e^h d_{\Omega}([x], [y])$  for  $[y] \in W^u([x])$  and  $d_{\Omega}(\tilde{f}([x]), \tilde{f}([y])) = e^{-h} d_{\Omega}([x], [y])$  for  $[y] \in \Omega^s([x])$ .

*Proof.* The pseudometric  $d^u$  is preserved under  $G$ . Since  $d_{\Omega}^s$  is defined via  $d^u$ , it is also preserved. Furthermore, we have that  $d^u$  transforms according to Theorem 2.5.  $\square$

Note that since  $G$  acts via invertible isometries on  $(\tilde{\Omega}, d_{\Omega})$ , it acts via homeomorphisms on  $(\tilde{\Omega}, d_{\Omega})$  despite the fact that the metric topology may not coincide with the quotient topology.

**Lemma 5.17.** For  $\tilde{\Omega}$  and  $\Omega^s([x])$  we have

- a) the topology on  $\Omega^s([x])$  induced by the metric  $d_{\Omega}^s$  is weaker than the quotient topology on  $\Omega^s([x])$  inherited as the quotient  $\Omega^s([x]) = (\tilde{\Lambda} \cap W^s(x))/\sim$ ;

- b) the topology on  $\tilde{\Omega}$  induced by the metric  $d_{\Omega}$  is weaker than the quotient topology on  $\tilde{\Omega}$  inherited as the quotient  $\tilde{\Omega} = \tilde{\Lambda}/\sim$ .

Furthermore for  $\tilde{\Omega}$  and  $\Omega^s([x])$  endowed with their metric topologies

- c) the quotient map  $\tilde{\Lambda} \rightarrow \tilde{\Omega}$  is continuous;  
d)  $\tilde{\Omega}$  and  $\Omega^s([x])$  are Hausdorff;  
e) either  $\Omega^s([x])$  is perfect for all  $[x] \in \tilde{\Omega}$  or is a singleton for all  $[x] \in \tilde{\Omega}$ .

*Proof.* To see 5.17(a), fix a  $t > 0$  and let  $U := B_{d_{\Omega}}([x], t)$ . Then

$$U = \left\{ y \in W^s(x) \cap \tilde{\Lambda} \mid W^u(y) \cap W^s(x \pm_u r') \neq \emptyset \quad \forall 0 < r' < \frac{1}{t} \right\}.$$

Clearly  $U$  is open as a subset of  $W^s(x) \cap \tilde{\Lambda}$  since for any  $y \in U$  we may find an open product chart containing  $y$  and  $y \pm_u \frac{1}{t}$ . 5.17(b) then follows from 5.17(a), and 5.17(c) follows from 5.17(b). 5.17(d) follows since the topologies are metric.

To see 5.17(e) assume  $\Omega^s([x])$  is not perfect, hence contains an isolated point  $[z]$ . Then for all  $z' \in W^u(z)$  we have  $[z']$  is isolated in  $\Omega^s([z'])$ . Periodic points are dense in  $\Lambda$ , hence we may find some periodic  $q \in \Lambda$  so that  $[\tilde{q}] \in W^u([z])$  for some lift  $\tilde{q}$  of  $q$  in  $\tilde{\Lambda}$ . Furthermore, since  $W^s(q) \cap \Lambda$  is dense in  $\Lambda$  we may assume that if there is any  $[z'] \in W^u([z])$  so that  $\Omega^s([z']) \neq [z']$  then  $\Omega^s([\tilde{q}]) \neq [\tilde{q}]$ . Passing to an iterate of  $f$  and choosing an appropriate lift  $\tilde{f}: \tilde{\Lambda} \rightarrow \tilde{\Lambda}$  of  $f$  we may assume that  $\tilde{f}([\tilde{q}]) = [\tilde{q}]$ . Since  $\tilde{f}$  is conformally contracting on  $\Omega^s([\tilde{q}])$ , the assumption  $\Omega^s([\tilde{q}]) \neq [\tilde{q}]$  contradicts that  $[\tilde{q}]$  is isolated in  $\Omega^s([\tilde{q}])$ . Thus we conclude for every  $[z'] \in W^u([z])$  that  $\Omega^s([z']) = [z']$ . Furthermore, we must have  $\tilde{\Omega} = \mathcal{O}_G(W^u([z]))$ , hence we have  $\Omega^s([x])$  is a singleton for every  $[x] \in \tilde{\Omega}$ .  $\square$

The proof of Theorem 1.1 will follow from considering two cases. In the first case we will assume that  $G$  acts properly discontinuously on  $\tilde{\Omega}$  and deduce that  $\Lambda$  is expanding. We will then show that if  $G$  fails to act properly discontinuously on  $\tilde{\Omega}$  then  $\tilde{\Lambda}$  has a product structure which will be used to obtain a conjugacy between  $f|_{\Lambda}$  and an automorphism of a toral solenoid.

**5.2. Case 1:  $G$  acts properly discontinuously on  $\tilde{\Omega}$ .** The goal of this section is to prove the following proposition.

**Proposition 5.18.** *Suppose  $G$  acts properly discontinuously on  $\tilde{\Omega}$ . Then  $\Lambda$  is expanding.*

For  $\tilde{x} \in \tilde{\Lambda}$  denote by  $C^s(\tilde{x})$  the connected component of  $\tilde{\Lambda} \cap W^s(\tilde{x})$  containing  $\tilde{x}$ . We define  $r: \Lambda \rightarrow \mathbb{R} \cup \{\infty\}$  by

$$r: x \mapsto \sup_{\tilde{y} \in C^s(\tilde{x})} \{d^s(\tilde{x}, \tilde{y})\}$$

where  $\tilde{x}$  is any lift of  $x$  to  $\tilde{\Lambda}$ .

Given a metric space  $(X, \rho)$  and a subset  $Y \subset X$  we call

$$\text{diam}(Y) := \sup\{\rho(x, y) \mid x, y \in Y\}$$

the *diameter* of  $Y$ . For any  $x \in X$  and  $Y \subset X$  we write

$$\rho(x, Y) := \inf\{\rho(x, y) \mid y \in Y\}.$$

**Definition 5.19.** Let  $\{A_n\}$  be a countable sequence of subsets in a metric space  $(X, \rho)$ . We define the *Kuratowski limit supremum* by

$$\overline{\lim}_{n \rightarrow \infty} A_n := \{x \in X \mid \liminf_{n \rightarrow \infty} \rho(x, A_n) = 0\}.$$

Clearly the Kuratowski limit supremum is a closed set for any collection  $\{A_n\}$ .

**Lemma 5.20.** *Let  $(X, \rho)$  be a proper metric space; that is, one for which any closed ball  $\{y \in X \mid d(x, y) \leq R\}$  is compact. Let  $\{A_n\} \subset X$  be a countable sequence of subsets such that*

- (1)  $\overline{\lim}_{n \rightarrow \infty} \text{diam}(A_n)$  is finite;
- (2) each  $A_n$  is connected;
- (3) there exists a Cauchy sequence  $\{x_n\}$  with  $x_n \in A_n$ .

*Then  $\overline{\lim}_{n \rightarrow \infty} A_n$  is connected.*

Note that the result need not hold if assumption 3 is omitted.

*Proof.* By assumption 3, we may fix  $x := \lim_{n \rightarrow \infty} x_n \in \overline{\lim}_{n \rightarrow \infty} A_n$ . By assumption 1 we may find an  $L$  and  $N$  so that for all  $n \geq N$ , the inclusion  $A_n \subset B(x, L)$  holds. Suppose  $\overline{\lim}_{n \rightarrow \infty} A_n$  is disconnected. Let  $N_1, N_2 \subset B(x, L)$  be two disjoint open sets such that  $x \in N_1$ ,  $(\overline{\lim}_{n \rightarrow \infty} A_n) \subset N_1 \cup N_2$  and  $(\overline{\lim}_{n \rightarrow \infty} A_n) \cap N_2 \neq \emptyset$ .

By assumption 3 we may find an  $M$  such that for all  $n \geq M$ , we have  $A_n \cap N_1 \neq \emptyset$ . Furthermore, we may find an infinite subsequence  $\{n_j\}$  such that  $A_{n_j} \cap N_2 \neq \emptyset$ . Since  $A_{n_j}$  is connected we have  $A_{n_j} \cap \partial(N_1) \neq \emptyset$ . For each  $j \in \mathbb{N}$  pick some  $a_j \in A_{n_j} \cap \partial(N_1)$ . Then since  $\partial(N_1)$  is compact we may find some  $y \in \partial(N_1)$  that is an accumulation point of  $\{a_j\}$ . But this implies that  $y \in \overline{\lim}_{n \rightarrow \infty} A_n$  whence  $\overline{\lim}_{n \rightarrow \infty} A_n \cap \partial(N_1) \neq \emptyset$ , a contradiction.  $\square$

**Lemma 5.21.** *The function  $r: \Lambda \rightarrow \mathbb{R}^+ \cup \{\infty\}$  is upper semicontinuous.*

*Proof.* We prove the lemma for the pull-back of the function  $r: \tilde{\Lambda} \rightarrow \mathbb{R}^+ \cup \{\infty\}$ . Clearly the lemma holds at  $x \in \tilde{\Lambda}$  if  $r(x) = \infty$ . We assume otherwise.

The function  $r$  is clearly continuous along unstable leaves. Consequently, we need only show that for  $\tilde{x}_i \in W_\epsilon^s(x)$ , if  $\tilde{x}_i \rightarrow x$  then  $r(x) \geq \overline{\lim}_{i \rightarrow \infty} r(\tilde{x}_i)$ . Passing to a subsequence  $\{x_n\} \subset \{\tilde{x}_i\}$  we may assume that

$$\lim_{n \rightarrow \infty} r(x_n) = \overline{\lim}_{i \rightarrow \infty} r(\tilde{x}_i).$$

If  $r(x) < \infty$  but the lemma failed at  $x$ , we could find  $\epsilon > 0$  and  $K$  so that for all  $n > K$  we have  $r(x_n) > r(x) + \epsilon$  and  $d^s(x, x_n) < \epsilon/3$ . Let  $\tilde{C}^s(x_n)$  denote the connected component of  $C^s(x_n) \cap \overline{B_{d^s}(x, r(x) + \epsilon/3)}$  containing  $x_n$ . (Here  $B_{d^s}(x, R)$  denotes the  $d^s$ -ball in  $W^s(x)$  of radius  $R$ .)

Let  $\Xi = \overline{\lim}_{n \rightarrow \infty} \tilde{C}^s(x_n)$ . By Lemma 5.20,  $\Xi$  is connected and hence we have  $\Xi \subset C^s(x)$ . On the other hand, the assumption on  $r(x_n)$  ensures that  $\tilde{C}^s(x_n) \cap \partial(B_{d^s}(x, r(x) + \epsilon/3)) \neq \emptyset$  for all  $n \geq K$ . Hence  $\Xi \cap \partial(B_{d^s}(x, r(x) + \epsilon/3)) \neq \emptyset$ , contradicting the definition of  $r(x)$ .  $\square$

**Corollary 5.22.** *Either  $r \equiv 0$  or  $r \equiv \infty$ .*

*Proof.* Suppose first that the range of  $r$  does not contain  $\infty$ . Then by upper semicontinuity,  $r$  is globally bounded. Let  $M = \max\{r(x) \mid x \in \Lambda\}$ . By hyperbolicity of  $f$  on  $\Lambda$  and boundedness of  $r$  we find an  $m \in \mathbb{N}$  so that

$$f^m(\pi(C^s(\tilde{x}))) \subset W_\epsilon^s(f^m(x))$$

(where  $\tilde{x}$  is a lift of  $x$ ), hence  $r(f^{m+1}(x)) \leq \lambda r(f^m(x))$  for all  $x \in \Lambda$ . On the other hand, since  $f$  is a homeomorphism, we should have

$$\max\{r(f^m(x)) \mid x \in \Lambda\} = \max\{r(f^{m+1}(x)) \mid x \in \Lambda\}.$$

But then  $M = \lambda M$  which implies  $M = 0$ .

Now if  $r(x) \neq \infty$  then  $r(y) \neq \infty$  for all  $y \in W^u(x)$ . Indeed, let  $\tilde{x}$  be a lift of  $x$ ,  $\tilde{y}$  the lift of  $y$  contained in  $W^u(\tilde{x})$ , and  $L = l^u(\tilde{x}, \tilde{y})$ . Let  $\mathcal{U}$  be a cover of  $C^s(\tilde{y})$ . Then for every  $z \in C^s(\tilde{y})$  there is an  $\epsilon(z) > 0$  so that  $W_{\epsilon(z)}^s(z) \subset U$  for some  $U \in \mathcal{U}$  and the set

$$V(z) := \{z' +_u l \mid z' \in W_{\epsilon(z)}^s(z), |l| \leq |L|\}$$

is a product chart. But then  $\{V(z)\}$  covers  $C^s(\tilde{x})$ , whence we conclude that  $\mathcal{U}$  admits a finite subcover.

Thus if  $r(x) = \infty$  for some  $x \in \Lambda$ , then  $r(y) = \infty$  for all  $y \in W^u(x)$ . Since  $W^u(x)$  is dense in  $\Lambda$ , the upper semicontinuity of  $r$  implies  $r \equiv \infty$ .  $\square$

We thus establish that  $\Lambda$  is expanding under the assumption that  $G$  acts properly discontinuously on  $\tilde{\Omega}$ .

*Proof of Proposition 5.18.* Let  $\Omega = \tilde{\Omega}/G$  be the orbit space. Note that since  $G$  acts properly discontinuously,  $\Omega$  is Hausdorff. Denote the canonical projections by  $\pi: \tilde{\Lambda} \rightarrow \Lambda$ ,  $q: \tilde{\Lambda} \rightarrow \tilde{\Omega}$ ,  $\pi': \tilde{\Omega} \rightarrow \Omega$ . Consider the diagram

$$\begin{array}{ccc} \tilde{\Lambda} & \xrightarrow{q} & \tilde{\Omega} \\ \pi \downarrow & & \downarrow \pi' \\ \Lambda & & \Omega \end{array}$$

Since the equivalence classes of  $\sim$  are  $G$ -invariant, the  $G$ -orbit of  $q(y)$  is equivalent to the  $G$ -orbit of  $q(g(y))$  for any  $g \in G$  and  $y \in \tilde{\Lambda}$ . Thus we may find a map  $q'$  so that the diagram

$$\begin{array}{ccc} \tilde{\Lambda} & \xrightarrow{q} & \tilde{\Omega} \\ \pi \downarrow & & \downarrow \pi' \\ \Lambda & \xrightarrow{q'} & \Omega \end{array}$$

commutes.

Since  $\Lambda$  is compact and  $\Omega$  is Hausdorff,  $q'$  is proper, whence  $q$  is proper. Hence the equivalence classes of  $\sim$  must be compact subsets of  $\tilde{\Lambda}$ . By Claim 5.12(d) and Corollary 5.22 this implies  $r \equiv 0$ ; hence the connected components of  $\Lambda \cap W^s(x)$  are singletons and  $\Lambda$  is expanding.  $\square$

**5.3. Case 2:  $G$  fails to act properly discontinuously on  $\tilde{\Omega}$ .** In the case that  $G$  fails to act properly discontinuously at some point in  $\tilde{\Omega}$ , we show that  $\Lambda$  is homeomorphic to a toral solenoid and  $f|_{\Lambda}$  is conjugate to a solenoidal automorphism.

**5.3.1. Metric properties of  $\tilde{\Omega}$ .** We first enumerate some additional properties of the metric  $d_{\Omega}$  and the action of  $G$  on  $\tilde{\Omega}$ .

**Claim 5.23.** *The following hold in the metric space  $(\tilde{\Omega}, d_{\Omega})$ .*

- a) *Let  $d_{\Omega}([x], [y]) < 1$ . Then  $W^u(y) \cap W^s(x) \neq \emptyset$ .*

b) We have  $[z_j] \rightarrow [x]$  in  $(\tilde{\Omega}, d_\Omega)$  if and only if  $d^u([x], [z_j]) \rightarrow 0$  and

$$d_\Omega^s([x], [W^u(z_j) \cap W^s(x)]) \rightarrow 0.$$

Note we have  $W^u(z_j) \cap W^s(x) \neq \emptyset$  for sufficiently large  $j$  by [5.23\(a\)](#).

c) Fix  $L \in \mathbb{R}$ . If  $g_i([x]) \rightarrow [x]$  then  $g_i([x] +_u L) \rightarrow [x] +_u L$ .

*Proof.* Fix  $R > 1$  so that  $d_\Omega([x], [y]) < \frac{1}{R}$ . Let  $\xi = ([x_0], [y_0], \dots, [y_k]) \in \Xi([x], [y])$  be as in Definition [5.15](#) with  $l(\xi) < \frac{1}{R}$ . Then we clearly have  $d^u(x, x_j) < \frac{1}{R} < R$  for all  $0 \leq j \leq k$ . Since we must also have  $d_\Omega^s([x_j], [y_{j-1}]) < \frac{1}{R}$  for  $1 \leq j \leq k$ , we inductively see that  $W^u(y_j) \cap W^s(x) \neq \emptyset$ , for each  $0 \leq j \leq k$  hence [5.23\(a\)](#) holds.

For  $[x_i]$  and  $[y_i]$  as above, denote by  $H([y_i]) = H([x_i]) := [W^u(y_i) \cap W^s(x)] = [W^u(x_i) \cap W^s(x)]$ . We check that for each  $y_i$

$$(7) \quad d_\Omega^s([x], H([y_i])) \leq \frac{R}{R^2 - 1}.$$

Indeed since  $d_\Omega^s([x_j], [y_{j-1}]) < \frac{1}{R}$  then

$$R - \frac{1}{R} \leq r^s(H([x_j]), H([y_{j-1}]))$$

from which we obtain

$$d_\Omega^s(H([x_j]), H([y_{j-1}])) \leq \frac{R}{R^2 - 1}$$

for all  $1 \leq j \leq k$ . Furthermore,  $d_\Omega^s(H([x_j]), H([y_j])) = 0$  for all  $0 \leq j \leq k-1$ , hence applying [\(6\)](#) recursively one obtains [\(7\)](#). In particular

$$(8) \quad d_\Omega^s([x], H([y])) \leq \frac{R}{R^2 - 1}.$$

Hence, by setting  $y = z_j$  and letting  $R \rightarrow \infty$  in [\(8\)](#), we see that  $d_\Omega([z_j], [x]) \rightarrow 0$  implies  $d^u([z_j], [x]) \rightarrow 0$  and

$$d_\Omega^s([x], H([z_j])) \rightarrow 0.$$

Furthermore, we clearly have  $d^u([x], [z_j]) \rightarrow 0$  and  $d_\Omega^s([x], H([z_j])) \rightarrow 0$  implies that  $d_\Omega([x], [z_j]) \rightarrow 0$  hence both implications in [5.23\(b\)](#) follow.

Note that  $g_i(x +_u L) = g_i(x) \pm_u L$  depending of whether  $g_i$  preserves the transverse orientation on  $\mathcal{W}^s$ . However for  $g$  such that  $l^u([x], g([x])) = t$  and

$$d_\Omega^s([x], [W^u(g(x)) \cap W^s(x)]) < \frac{1}{|t|}$$

$g$  can not reverse the orientation since otherwise we would have

$$W^s(x +_u t/2) \cap W^s(g(x +_u t/2)) = W^s(x +_u t/2) \cap W^s(g(x) -_u t/2) \neq \emptyset,$$

a contradiction unless  $g$  is the identity. Thus we may assume that for  $g_i$  in [5.23\(c\)](#),  $g_i(x +_u L) = g_i(x) +_u L$ .

Now let  $L \in \mathbb{R}$  be given. By forgetting initial terms and invoking [5.23\(a\)](#) and [5.23\(b\)](#), we may assume that for all  $i$

$$H([g_i(x)]) := [W^u(g_i(x)) \cap W^s(x)]$$

is defined, and  $d_\Omega^s([x], H([g_i(x)])) < \frac{1}{|L|+1}$ . Then by definition of  $r^s$  we have  $W^u(g_i(x)) \cap W^s(x +_u L) \neq \emptyset$ , hence  $H([g_i(x)]) +_u L = [W^u(g_i(x)) \cap W^s(x +_u L)]$ . As above we have

$$r^s([x], H([g_i(x)])) - L \leq r^s([x +_u L], H([g_i(x)] +_u L))$$

hence

$$d_\Omega^s([x +_u L], H([g_i(x)] +_u L)) \leq \frac{d_\Omega^s([x], H([g_i(x)]))}{1 - L \cdot d_\Omega^s([x], H([g_i(x)]))}$$

which by 5.23(b), establishes 5.23(c).  $\square$

Given a subset  $S \subset G$  we say  $S$  acts *properly discontinuously* at  $[x]$  if there is some open set  $U \ni [x]$  so that  $s(U) \cap U \neq \emptyset$  implies  $s = 1$  for any  $s \in S$ . Since  $G$  acts freely on  $\tilde{\Omega}$ , every finite subset  $S \subset G$  acts properly discontinuously at every point of  $\tilde{\Omega}$ .

**Lemma 5.24.** *Suppose a set  $S \subset G$  acts properly discontinuously at one point  $[x] \in \tilde{\Omega}$ . Then  $S$  acts properly discontinuously at every point  $[y] \in \tilde{\Omega}$ .*

*Proof.* Let  $U \subset \tilde{\Omega}$  be an open neighborhood of  $[x]$  so that for each  $s \in S$ , we have  $s(U) \cap U \neq \emptyset$  implies  $s = 1$ . Then  $S$  acts properly discontinuously at every point  $[y] \in U$ . By Claim 5.23(c), if  $S$  acts properly discontinuously at  $[y]$  then it acts properly discontinuously at every point in  $W^u([y])$ , hence we have that  $S$  acts properly discontinuously at every point of  $W^u(U)$ . But then  $S$  acts properly discontinuously at every point of  $\mathcal{O}_G(W^u(U)) = \tilde{\Omega}$ .  $\square$

Setting  $S = G$ , we have the contrapositive.

**Corollary 5.25.** *If  $G$  fails to act properly discontinuously at one point  $[x] \in \tilde{\Omega}$  then it fails to act properly discontinuously at every point  $[y] \in \tilde{\Omega}$ .*

Furthermore we have

**Corollary 5.26.** *Let  $g_i([x]) \rightarrow [x]$  in  $(\tilde{\Omega}, d_\Omega)$  for some sequence  $\{g_i\} \subset G$ . Then  $g_i([y]) \rightarrow [y]$  in  $(\tilde{\Omega}, d_\Omega)$  for any  $[y] \in \tilde{\Omega}$ .*

*Proof.* Set  $S = \{g_i\}$ . If  $g_i([y])$  fails to converge to  $[y]$  then there is a neighborhood  $U$  of  $[y]$  and an infinite subset  $S' \subset S$  so that  $s(U) \cap U = \emptyset$  for all  $s \in S'$ . But then  $S'$  acts properly discontinuously at  $[y]$  which by Lemma 5.24 implies that  $S'$  acts properly discontinuously at  $[x]$ . But  $S'$  corresponds to a infinite subsequence of  $\{g_i\}$ , contradicting that  $g_i([x]) \rightarrow [x]$ .  $\square$

We now show that the above convergence happens uniformly in  $[y]$ . Define a map  $\zeta: G \times \tilde{\Omega} \rightarrow [0, \infty)$  by

$$\zeta: (g, [x]) \mapsto d_\Omega([x], g([x])).$$

Endowing  $G$  with the discrete topology, we have that  $\zeta$  is continuous. Since the metric topology on  $\tilde{\Omega}$  is weaker than the quotient topology, the quotient map  $q: \tilde{\Lambda} \rightarrow \tilde{\Omega}$  induces a continuous map  $q^*\zeta: G \times \tilde{\Lambda} \rightarrow [0, \infty)$ . Now

$$q^*\zeta(g, x) = q^*\zeta(g, g'(x))$$

for all  $g, g' \in G$  hence  $q^*\zeta$  induces a continuous map  $\bar{\zeta}: G \times \Lambda \rightarrow [0, \infty)$ .

As a result we have,

**Lemma 5.27.** *Assume  $g_i([x]) \rightarrow [x]$  for some  $[x] \in \tilde{\Omega}$ . Then given an  $\epsilon > 0$ , we may find an  $N$  so that for all  $i \geq N$  and  $[y] \in \tilde{\Omega}$  we have  $d_\Omega([y], g_i([y])) < \epsilon$ .*

*Proof.* Assume the conclusion fails for some fixed  $\epsilon$ . Passing to an infinite subsequence, we may assume the conclusion fails for all  $i \in \mathbb{N}$ . Let  $\{y_i\} \subset \Lambda$  be such that  $\bar{\zeta}(g_i, y_i) > \epsilon$  and, again passing to a subsequence, let  $z \in \Lambda$  be a limit point of  $\{y_j\}$ . Let  $\tilde{z}$  be a lift of  $z$  and  $\{\tilde{y}_i\}$  a lift of  $\{y_i\}$  such that  $\tilde{z}$  is a limit point of  $\{\tilde{y}_i\}$ . Because the metric topology on  $\tilde{\Omega}$  is weaker than the quotient topology, we have that  $[\tilde{z}]$  is a limit point of the sequence  $\{[\tilde{y}_i]\}$  with respect to the metric topology.

Passing to a subsequence we may assume  $d_\Omega([\tilde{y}_i], [\tilde{z}]) < \epsilon/4$  for all  $i$  from which we obtain

$$d_\Omega([\tilde{z}], g_i([\tilde{z}])) \geq d_\Omega([\tilde{y}_i], g_i([\tilde{y}_i])) - d_\Omega([\tilde{z}], [\tilde{y}_i]) - d_\Omega(g_i([\tilde{y}_i]), g_i([\tilde{z}]))$$

hence  $d_\Omega([\tilde{z}], g_i([\tilde{z}])) > \epsilon/2$  for all  $i$ , contradicting Corollary 5.26.  $\square$

**5.3.2. Global product structure.** We now establish that when  $G$  fails to act properly discontinuously, the set  $\tilde{\Lambda}$  has a *global product structure*; that is, for all  $x, y \in \tilde{\Lambda}$  we have  $W^u(x) \cap W^s(y) \neq \emptyset$ .

**Lemma 5.28.** *Let  $G$  fail to act properly discontinuously on  $\tilde{\Omega}$ . Then  $\Omega^s([x])$  is a singleton for all  $x$  and  $\tilde{\Lambda} = W^u([x])$  for any  $x \in \tilde{\Lambda}$ .*

*Proof.* Fix some  $R > 1$ , and choose an  $R' > R$  with the property that  $\frac{R'}{(R')^2 - 1} \leq \frac{1}{R}$ .

Suppose  $G$  fails to act properly discontinuously at  $[x]$ . Then we may find a subset  $\{g_i\} \subset G$  so that  $g_i([x]) \rightarrow [x]$  and  $d_\Omega([x], g_i([x])) < 1/R$  for all  $i$ . Let  $N = B_{d_\Omega^s}([x], \frac{1}{R})$ . As guaranteed by Lemma 5.27, we may remove initial terms of  $\{g_i\}$  so that  $d_\Omega([y], g_i([y])) \leq \frac{1}{R'}$  for all  $i$  and  $[y] \in \tilde{\Omega}$ .

For  $[y] \in N$  define

$$H_{g_i}([y]) := [W^u(g_i(y)) \cap W^s(x)].$$

Then, as in (8) we have

$$d_\Omega^s([y], H_{g_i}([y])) \leq \frac{R'}{(R')^2 - 1} \leq \frac{1}{R}.$$

But then by (6) we have

$$d_\Omega^s([x], H_{g_i}([y])) \leq \max\{d_\Omega^s([x], [y]), d_\Omega^s([y], H_{g_i}([y]))\} \leq \frac{1}{R}$$

hence  $H_{g_i}(N) \subset N$ . Furthermore  $H_{g_i^{-1}}$  is defined on  $N$  and by the same argument as above  $H_{g_i^{-1}}(N) \subset N$ . Hence  $H_{g_i}(N) = N$ . In particular, setting  $L = l^u(x, g_i(x))$  we have

$$g_i(N) = N +_u L.$$

Set

$$D = \{y +_u l \mid |l| < R, [y] \in N\}.$$

Since  $d^u([x], g_i([x])) < \frac{1}{R} < R$ , we have  $g_i(N) \subset D$  and  $g_i^{-1}(N) \subset D$ . Inductively, we see that for any  $k$  and

$$[x], [y] \in \bigcup_{|j| \leq k} g_i^j(D)$$



that  $W^u(x) \cap W^s(y) \neq \emptyset$ , hence  $\bigcup_{|j| \leq k} g_i^j(D)$  is a product chart. In particular, we have equality between product charts  $\bigcup_{j \in \mathbb{Z}} g_i^j(D) = W^u([x])$ , thus showing that  $[x]$  is isolated in  $\Omega^s([x])$ . By Lemma 5.17(e) we see that  $\Omega^s([x]) = [x]$  for all  $[x] \in \tilde{\Omega}$ .

Considering  $[x]$  as a subset of  $\tilde{\Lambda}$  we have  $\tilde{\Lambda} = \mathcal{O}_G(W^u([x]))$  and

$$\tilde{B} = \mathcal{O}_G(W^s(W^u([x]))).$$

If  $g \in G$  is such that  $g(W^u([x])) \neq W^u([x])$  then

$$g(W^s(W^u([x]))) \cap (W^s(W^u([x]))) = \emptyset.$$

Thus we must have  $\tilde{\Lambda} = W^u([x])$  since otherwise  $\tilde{B}$  would not be connected.  $\square$

The following is immediate from Lemma 5.28.

**Corollary 5.29.** *When  $G$  fails to act properly discontinuously on  $\tilde{\Omega}$  then  $\tilde{\Lambda}$  admits a global product structure.*

We now shift out attention back to  $\tilde{\Lambda}$ , under the assumption that  $\tilde{\Lambda}$  admits a global product structure. Our objective is to prove the following.

**Proposition 5.30.** *Assume  $\tilde{\Lambda}$  has a global product structure. Then  $f: \Lambda \rightarrow \Lambda$  is conjugate to a leaf-wise hyperbolic automorphism of a toral solenoid (see Section 4).*

To prove Proposition 5.30 we need the following technical result. We note that the proof technique for Proposition 5.30, including Lemma 5.31, are adapted from [9].

**5.3.3. Global shadowing lemma.** Let  $(\Upsilon, \tau)$  be a metrizable topological space. For a fixed  $k$  let  $\rho$  be the standard metric on  $\mathbb{R}^k$ . Furthermore let  $\{d_x\}_{x \in \mathbb{R}^k}$  be a family of complete metrics on  $\Upsilon$  (each inducing the topology  $\tau$ ) such that

- (1)  $d_x$ -balls in  $\Upsilon$  are precompact for all  $x \in \mathbb{R}^k$
- (2) the induced map  $\mathbb{R}^k \times \Upsilon \times \Upsilon \rightarrow \mathbb{R}$  given by

$$(x, \xi, \eta) \mapsto d_x(\xi, \eta)$$

is continuous.

Let  $\Omega = \mathbb{R}^k \times \Upsilon$  with projections  $\pi_1: \Omega \rightarrow \mathbb{R}^k$  and  $\pi_2: \Omega \rightarrow \Upsilon$ . Given  $x, y \in \Omega$  let  $\Xi(x, y)$  be the set of sequences  $\{x_0, y_0, \dots, x_k, y_k\}$  such that

- (1)  $x = x_0$  and  $y = y_k$ ;
- (2)  $\pi_1(x_j) = \pi_1(y_j)$  for all  $0 \leq j \leq k$ ;
- (3)  $\pi_2(x_j) = \pi_2(y_{j-1})$  for all  $1 \leq j \leq k$ .

Given an  $\xi \in \Xi(x, y)$  define

$$l(\xi) := \sum_{j=0}^k d_{\pi_1(x_j)}(\pi_2(x_j), \pi_2(y_j)) + \sum_{j=1}^k \rho(\pi_1(x_j), \pi_1(y_{j-1}))$$

and define

$$(9) \quad d(x, y) := \inf_{\xi \in \Xi(x, y)} \{l(\xi)\}.$$

Clearly  $d$  defines a metric on  $\Omega$ ; furthermore, the continuity of the function  $(x, \xi, \eta) \mapsto d_x(\xi, \eta)$  guarantees that the metric topology is consistent with the product topology of  $\mathbb{R}^k \times \Upsilon$ .

Given a metric space  $(X, d)$ , a homeomorphism  $f: X \rightarrow X$  is called *expanding* if there is some  $\mu > 1$  so that for all  $x, y \in X$ ,  $d(f(x), f(y)) \geq \mu d(x, y)$ . A sequence  $\{x_j\}_{j \in \mathbb{Z}} \subset (X, d)$  is called an *L-pseudo orbit* for  $f$  if  $d(f(x_j), x_{j+1}) \leq L$  for all  $j$ . Given an *L-pseudo orbit*  $\{x_j\}$  we say a point  $x \in X$  *shadows*  $\{x_j\}$  if there is some  $\delta$  so that  $d(f^j(x), x_j) \leq \delta$  for all  $j$ .

**Lemma 5.31** (Global Shadowing). *Let  $h: \Omega \rightarrow \Omega$  be a product homeomorphism  $h: (x, \xi) \mapsto (h_1(x), h_2(\xi))$ . Assume that  $h_1: \mathbb{R}^k \rightarrow \mathbb{R}^k$  is expanding with respect to the metric  $\rho$ . Furthermore assume that  $h_2: \Upsilon \rightarrow \Upsilon$  is asymptotically exponentially contracting on bounded sets with respect to each metric  $d_x$ ; that is, given an  $R > 0$ ,  $\xi \in \Upsilon$ , and  $x \in \mathbb{R}^k$  there are  $c > 0$  and  $\lambda < 1$ , depending continuously on  $(x, \xi) \in \Omega$ , so that if  $d_x(\xi, \zeta) \leq R$  then*

$$d_{h_1^j(x)}(h_2^j(\xi), h_2^j(\zeta)) \leq c\lambda^j d_x(\xi, \zeta)$$

for all  $j \geq 0$ . Additionally assume that  $\Omega$  admits a properly discontinuous action by a subgroup  $G$  of the group of isometries of  $(\Omega, d)$  such that  $h$  preserves  $G$ -orbits, and the quotient  $\Omega/G$  is compact. We have the following.

- a) Given a  $C > 0$  there is a  $K > 0$  so that if  $d(x, y) \leq C$  for any  $x, y \in \Omega$ , then

$$d_{\pi_1(y)}(\pi_2(y), \pi_2(x)) \leq K.$$

- b) Given an  $R > 0$  there are  $c > 0$  and  $\lambda < 1$  (depending only on  $R$ ) so that for any  $\xi, \zeta \in \Upsilon$  and  $x \in \mathbb{R}^k$ ,  $d_x(\xi, \zeta) \leq R$  implies

$$d_{h_1^j(x)}(h_2^j(\xi), h_2^j(\zeta)) \leq c\lambda^j d_x(\xi, \zeta).$$

- c) Given any *L-pseudo orbit*  $\{x_j\} \subset \Omega$  in the metric  $d$ , there is a point  $x \in \Omega$  that shadows  $\{x_j\}$ .  
d) Fix  $C > 0$ . Then there is a sequence  $\{\epsilon_N\}$  with  $\epsilon_N \rightarrow 0$  as  $N \rightarrow \infty$  such that

$$d(h^j(x), h^j(y)) \leq C \quad \text{for } |j| \leq N$$

implies  $d(x, y) \leq \epsilon_N$ .

**Remark 5.32.** Note that Lemma 5.31 applies to  $\tilde{\Lambda}$  in the case that  $\tilde{\Lambda}$  has global product structure by choosing an  $x \in \tilde{\Lambda}$  and taking  $\Upsilon = \tilde{\Lambda} \cap W^s(x)$ ,  $k = 1$ ,  $\rho = d^u$  in Definition 5.8, and  $d_x$  the metrics  $d_x^s$  in Definition 5.1. Then the metric in Definition 5.9 corresponds to (9).

Furthermore, assuming the linear map  $A: \mathbb{R}^n \rightarrow \mathbb{R}^n$  is hyperbolic, we have that the cover  $\tilde{\mathcal{S}}$  constructed in Section 4 and endowed with the metric  $\tilde{d}$  constructed in Section 4.3 satisfies the hypotheses of Lemma 5.31 with  $\Upsilon = E^- \times \tilde{\Sigma}$  and  $k = \dim E^u$ .

*Proof of Lemma 5.31(a) and 5.31(b).* The hypotheses of Lemma 5.31 guarantee that the numbers  $K$ ,  $c$ , and  $\lambda$  can be chosen pointwise on  $\Omega$ . Since the group  $G$  acts via isometries, we may assume they are constant on  $G$ -orbits. Since the quotient  $\Omega/G$  is compact, we may choose uniform  $K$ ,  $c$ , and  $\lambda$ .  $\square$

*Proof of Lemma 5.31(c).* Because  $h_1$  is an expanding homeomorphism, there is a unique fixed point  $p \in \mathbb{R}^k$ . If  $\{x_j\}$  is an *L-pseudo orbit* for  $h$  in the metric  $d$  then  $\pi_1(x_j)$  is an *L-pseudo orbit* for  $h_1$  in the metric  $\rho$ ; that is,  $\rho(\pi_1(h(x_j)), \pi_1(x_{j+1})) \leq L$ . But then

$$\rho(\pi_1(h^{-j}(x_j)), \pi_1(h^{-j-1}(x_j + 1))) \leq \mu^{-j-1} L$$

from which we see that the sequence

$$\{\pi_1(x_0), \pi_1(h^{-1}(x_1)), \dots, \pi_1(h^{-j}(x_j)), \dots\}$$

is Cauchy. Set

$$y = \lim_{j \rightarrow \infty} \{\pi_1(x_0), \pi_1(h^{-1}(x_1)), \dots, \pi_1(h^{-j}(x_j)), \dots\}.$$

Let  $g: \mathbb{R} \rightarrow \mathbb{R}$  be the map  $g: x \mapsto \mu^{-1}(x + L)$ . Then  $g$  is contracting, hence has a unique fixed point. In particular the sequence  $\{x, g(x), g^2(x), \dots\}$  is bounded for any  $x$ . Set

$$R := \sup\{\rho(\pi_1(x_0), p), g(\rho(\pi_1(x_0), p)), g^2(\rho(\pi_1(x_0), p)), \dots\} < \infty.$$

Then for every  $j \leq 0$  we have

$$\rho(\pi_1(x_j), p) \leq R < \infty.$$

Taking  $C = R + L + \mu R$  we have that  $d\left((p, \pi_2(x_j)), (p, \pi_2(h(x_{j-1})))\right) \leq C$  for all  $j \leq 0$ , whence from [5.31\(a\)](#) we may find a  $K < \infty$  so that  $d_p(\pi_2(x_j), \pi_2(h(x_{j-1}))) \leq K < \infty$  for all  $j \leq 0$ . Thus from [5.31\(b\)](#) the sequence

$$\{\pi_2(x_0), \pi_2(h(x_{-1})), \pi_2(h^2(x_{-2})), \dots, \pi_2(h^j(x_{-j})), \dots\}$$

is Cauchy in the metric  $d_p$ . Let

$$z = \lim_{j \rightarrow \infty} \{\pi_2(x_0), \pi_2(h(x_{-1})), \pi_2(h^2(x_{-2})), \dots, \pi_2(h^j(x_{-j})), \dots\}.$$

Then one easily verifies that  $x := (y, z)$  shadows the  $L$ -pseudo orbit  $\{x_j\}$ .  $\square$

*Proof of Lemma [5.31\(d\)](#).* For  $C$ , let  $K$  be as in [5.31\(a\)](#) and let  $c, \lambda$  be as in [5.31\(b\)](#) with  $R = K$ . Fix  $x, y \in \Omega$  so that  $d(h^j(x), h^j(y)) \leq C$  for  $|j| \leq N$ . Then we have

$$\rho(\pi_1(h^j(x)), \pi_1(h^j(y))) \leq C$$

for  $j \leq N$ , hence

$$\rho(\pi_1(x), \pi_1(y)) \leq \mu^{-N} C.$$

Furthermore,

$$d_{h_1^{-N}(\pi_1(y))} \left( h_2^{-N}(\pi_2(y)), h_2^{-N}(\pi_2(x)) \right) \leq K$$

hence

$$d_y(\pi_2(y), \pi_2(x)) \leq cK\lambda^N$$

from which we conclude that

$$d(x, y) \leq cK\lambda^N + \mu^{-N} C$$

and the conclusion follows with  $\epsilon_N := cK\lambda^N + \mu^{-N} C$ .  $\square$

5.3.4. *Proof of Proposition 5.30.* We now return to the proof of Proposition 5.30. We have the following observation.

**Lemma 5.33.** *When  $\tilde{\Lambda}$  has global product structure the covering group  $G := \pi_1(B)$  is torsion-free abelian.*

*Proof.* Fix any  $x \in \tilde{\Lambda}$ . Since  $\tilde{\Lambda}$  has global product structure, we may canonically identify  $\tilde{\Lambda}$  with  $W^u(x) \times (\tilde{\Lambda} \cap W^s(x))$ . Let  $\sim_s$  be the equivalence relation  $z \sim_s y$  if  $z \in W^s(y)$ . Then we have a canonical identification of  $W^u(x)$  with  $\Lambda/\sim_s$  which induces a  $G$ -action on  $W^u(x)$ . By Theorem 2.5 and the construction of the pseudo-metric  $d^u$  on  $\tilde{\Lambda}$ , we have that  $G$  acts on  $(W^u(x), d^u)$  via isometries. Furthermore the isometries are orientation-preserving since otherwise there would be a non-identity  $g \in G$  and  $y \in W^u(x)$  with  $W^s(y) = W^s(g(y))$ . Hence we naturally identify  $G$  with a subgroup of the orientation-preserving isometries of  $\mathbb{R}$  and the result follows.  $\square$

Note that  $G$  need not be finitely generated. However, we can represent  $G$  as the limit of a directed system of finitely generated, torsion-free abelian groups as follows. Let  $\{\mathcal{R}_j\}$  be a Markov partition of  $\Lambda$  and let  $\{\tilde{\mathcal{R}}_{j,\alpha}\}_{\alpha \in G}$  be a lift of the Markov partition where each  $\tilde{\mathcal{R}}_{j,\alpha}$  is homeomorphic to  $\mathcal{R}_j$ . Fix an  $x \in \tilde{\Lambda}$ . Recall that  $W^u(\pi(x))$  is dense in  $\Lambda$ . For each  $j$ , distinguish a  $\overline{\mathcal{R}}_j \in \{\tilde{\mathcal{R}}_{j,\alpha}\}$  so that

$$(10) \quad W^u(x) \cap \text{int}(\overline{\mathcal{R}}_j) \neq \emptyset.$$

Set  $D = \bigcup_j \overline{\mathcal{R}}_j$ . Then  $D$  is a fundamental domain for the covering  $\tilde{\Lambda} \rightarrow \Lambda$ .

Let  $H \subset G$  be the subgroup generated by  $\{\alpha \in G \mid \alpha(D) \cap D \neq \emptyset\}$ . Since  $D$  is compact and  $G$  acts discontinuously,  $H$  is finitely generated. Let  $N := \mathcal{O}_H(D)$ . Note  $N$  is clopen in  $\tilde{\Lambda}$  and  $H = \{\alpha \in G \mid \alpha(N) = N\}$ .

**Claim 5.34.** *There is a lift  $\tilde{f}$  of  $f$  so that  $\tilde{f}(N) \subset N$ . Indeed  $f_*H \subset H$ , where  $f_*: G \rightarrow G$  is the automorphism induced by the diffeomorphism  $f: B \rightarrow B$ .*

*Proof.* Since  $D$  is a lift of a Markov partition, by the definition of  $H$  if  $y \in N$  then  $W^u(y) \subset N$ . Choose a lift  $\tilde{f}$  of  $f: \Lambda \rightarrow \Lambda$  so that

$$(11) \quad \tilde{f}(x) \cap N \neq \emptyset$$

where  $x$  is as chosen above.

Now  $\tilde{f}(N) = \mathcal{O}_{f_*H}(\tilde{f}(D))$ . We note that  $f_*H$  is the subgroup of  $G$  generated by the set  $\mathcal{A} := \{\alpha \in G \mid \alpha(\tilde{f}(D)) \cap \tilde{f}(D) \neq \emptyset\}$ . By (10) and (11) we have that  $\text{int}(\tilde{f}(\overline{\mathcal{R}}_j)) \cap N \neq \emptyset$ , hence by the Markov property (Definition 2.2(3)) we have  $\tilde{f}(\overline{\mathcal{R}}_j) \subset N$  for each  $j$ . In particular  $\tilde{f}(D) \subset N$ . Hence we conclude that  $\mathcal{A} \subset H$ ,  $f_*H \subset H$  and  $\tilde{f}(N) \subset N$ .  $\square$

Note that  $N$  is a covering of  $\Lambda$  with covering group  $H$ . Also, for any  $y \in \tilde{\Lambda}$  and  $\tilde{f}$  as in Claim 5.34 there is some  $m$  so that  $\tilde{f}^m(y) \in N$ . Consequently, we may reconstruct  $\tilde{\Lambda}$  and the covering group  $G$  as limits of the directed systems

$$\Lambda \cong \varinjlim \left\{ N \xrightarrow{\tilde{f}} N \xrightarrow{\tilde{f}} N \xrightarrow{\tilde{f}} \dots \right\}$$

and

$$G \cong \varinjlim \left\{ H \xrightarrow{f_*} H \xrightarrow{f_*} H \xrightarrow{f_*} \dots \right\}.$$

Fix an isomorphism  $\Phi: H \rightarrow \mathbb{Z}^k$  and let  $A: \mathbb{Z}^k \rightarrow \mathbb{Z}^k$  be the endomorphism  $\Phi \circ f_* \circ \Phi^{-1}$ . Considering  $\mathbb{Z}^k$  as embedded in  $\mathbb{R}^k$ ,  $A: \mathbb{Z}^k \rightarrow \mathbb{Z}^k$  induces a linear automorphism on  $\mathbb{R}^k$  and a surjective endomorphism on the quotient  $\mathbb{T}^k = \mathbb{R}^k / \mathbb{Z}^k$ , also denoted by  $A$ . Let  $\mathcal{S}_A$  and  $\tilde{\mathcal{S}}$  be the solenoid and its cover constructed in Section 4, and let  $\sigma_A$  and  $\tilde{\sigma}$  be the respective shift automorphisms.

Fix an identification  $G = \varinjlim (H, f_*)$  whence  $f_*: G \rightarrow G$  is identified with the shift map  $\tau_{f_*}|_G$ . We have that the diagram

$$\begin{array}{ccc} H & \xrightarrow{f_*} & H \\ \Phi \downarrow & & \downarrow \Phi \\ \mathbb{Z}^k & \xrightarrow{A} & \mathbb{Z}^k \end{array}$$

commutes, hence

$$\begin{array}{ccccccc} H & \xrightarrow{f_*} & H & \xrightarrow{f_*} & H & \xrightarrow{f_*} & H & \xrightarrow{f_*} & \dots \\ \Phi \downarrow & & \downarrow \Phi & & \downarrow \Phi & & \downarrow \Phi & & \\ \mathbb{Z}^k & \xrightarrow{A} & \mathbb{Z}^k & \xrightarrow{A} & \mathbb{Z}^k & \xrightarrow{A} & \mathbb{Z}^k & \xrightarrow{A} & \dots \end{array}$$

induces an isomorphism  $\tilde{\Phi}: G \rightarrow \mathbb{Z}^k[A^{-1}]$ . Furthermore, we have  $\tilde{\Phi} \circ f_* \circ \tilde{\Phi}^{-1} = \tau_A$  where  $\tau_A$  is as constructed in Section 4.2.

*Proof of Proposition 5.30.* Fix a lift  $\tilde{f}$  of  $f$ . Let  $\mathcal{S}_A, \sigma_A, \tilde{\mathcal{S}}$ , and  $\tilde{\sigma}$  be as above. Let  $D$  be a fundamental domain for the cover  $\tilde{\mathcal{S}} \rightarrow \mathcal{S}_A$ ; note that  $D$  will be compact. Let  $\tilde{\Phi}$  be the isomorphism between  $G$  and  $\mathbb{Z}^k[A^{-1}]$  above. We let  $\mathbb{Z}^k[A^{-1}]$  act by addition on  $\tilde{\mathcal{S}}$  via the action  $\tilde{\alpha}$  in Section 4.2.

Given a  $\xi \in \tilde{\mathcal{S}}$  we may find a sequence  $\{\alpha_j\} \subset \mathbb{Z}^k[A^{-1}]$  so that

$$(12) \quad \tilde{\sigma}^j(\xi) \in D + \alpha_j.$$

**Claim 5.35.** *There is an  $L$  so that for any  $x \in \tilde{\Lambda}$ ,  $\xi \in \tilde{\mathcal{S}}$ , and a sequence  $\{\alpha_j\}$  satisfying (12), the sequence  $\{(\tilde{\Phi}^{-1}(\alpha_j))(\tilde{f}^j(x))\}$  is an  $L$ -pseudo orbit.*

*Proof.* Let

$$\mathcal{A} = \{a \in \mathbb{Z}^k[A^{-1}] \mid (D + a) \cap \tilde{\sigma}(D) \neq \emptyset\}.$$

By Proposition 4.4(d) if  $\xi \in D + \alpha$  then  $\tilde{\sigma}(\xi) \in (D + a) + \tau_A(\alpha)$  for some  $a \in \mathcal{A}$ . In particular,  $\alpha_{j+1} = \tau_A \alpha_j + a$  for some  $a \in \mathcal{A}$ . We set

$$L = \sup\{d(\tilde{y}, (\tilde{\Phi}^{-1}(a))(\tilde{y})) \mid y \in \Lambda, a \in \mathcal{A}\}$$

where  $\tilde{y}$  is an arbitrary lift of  $y$  to  $\tilde{\Lambda}$ . Finiteness of  $\mathcal{A}$  guarantees  $L < \infty$ . Hence

$$\begin{aligned} & d\left(\tilde{f}\left((\tilde{\Phi}^{-1}(\alpha_j))(\tilde{f}^j(x))\right), (\tilde{\Phi}^{-1}(\alpha_{j+1}))(\tilde{f}^{j+1}(x))\right) \\ &= d\left((f_*(\tilde{\Phi}^{-1}(\alpha_j)))(\tilde{f}^{j+1}(x)), (\tilde{\Phi}^{-1}(\alpha_{j+1}))(\tilde{f}^{j+1}(x))\right) \\ &= d\left((\tilde{\Phi}^{-1}(\tau_A(\alpha_j)))(\tilde{f}^{j+1}(x)), (\tilde{\Phi}^{-1}(\alpha_{j+1}))(\tilde{f}^{j+1}(x))\right) \\ &\leq \max_{a \in \mathcal{A}} \left\{ d\left((\tilde{\Phi}^{-1}(\tau_A(\alpha_j)))(\tilde{f}^{j+1}(x)), (\tilde{\Phi}^{-1}(\tau_A(\alpha_j) + a))(\tilde{f}^{j+1}(x))\right) \right\} \\ &\leq L. \end{aligned}$$

Hence the claim holds.  $\square$

We define a map  $\Psi: \tilde{\mathcal{S}} \rightarrow \tilde{\Lambda}$  as follows. Fix a  $p \in \tilde{\Lambda}$ . Given  $\xi \in \tilde{\mathcal{S}}$  choose a sequence  $\{\alpha_j\} \subset \mathbb{Z}^k[A^{-1}]$  satisfying (12). Then define  $\Psi(\xi)$  to be the unique point  $x$  in  $\tilde{\Lambda}$  that shadows the  $L$ -pseudo orbit  $\{(\tilde{\Phi}^{-1}(\alpha_j))(\tilde{f}^j(p))\}$ . Note that Lemma 5.31(c) guarantees the point  $x$  exists, whereas Lemma 5.31(d) guarantees that the point  $x$  is unique. Furthermore, Lemma 5.31(d) guarantees that  $\Psi: \tilde{\mathcal{S}} \rightarrow \tilde{\Lambda}$  is continuous.

**Claim 5.36.**  $\Psi$  is proper.

*Proof.* For  $\xi \in \tilde{\mathcal{S}}$  and  $\alpha \in \mathbb{Z}^k[A^{-1}]$  we clearly have  $\Psi(\xi + \alpha) = (\tilde{\Phi}^{-1}(\alpha))(\Psi(\xi))$ , hence the map  $\Psi: \tilde{\mathcal{S}} \rightarrow \tilde{\Lambda}$  descends to a continuous map  $h: \mathcal{S}_A \rightarrow \Lambda$ .  $\square$

Since  $\Psi$  is proper, we have that  $A$  is hyperbolic. Thus as in Remark 5.32, Lemma 5.31 applies to  $\tilde{\mathcal{S}}$ . Hence, given a fundamental domain  $D \subset \tilde{\Lambda}$ , and  $x \in \tilde{\Lambda}$ , we choose  $\{g_i\}$  so that  $\tilde{f}^i(x) \in g_i(D)$ . Then as above we define  $\Psi'(x)$  to be the unique point  $\xi \in \tilde{\mathcal{S}}$  so that  $\xi$  shadows the pseudo orbit

$$\{\tilde{\Phi}(g_i)(e)\}$$

where  $e$  is the identity in  $\tilde{\mathcal{S}}$ . We thus obtain a map  $\Psi': \tilde{\Lambda} \rightarrow \tilde{\mathcal{S}}$ .

One easily verifies

- (1)  $\Psi$  and  $\Psi'$  are inverses;
- (2) the diagram

$$\begin{array}{ccc} \tilde{\mathcal{S}} & \xrightarrow{\tilde{\sigma}} & \tilde{\mathcal{S}} \\ \Psi \downarrow & & \downarrow \Psi \\ \tilde{\Lambda} & \xrightarrow{\tilde{f}} & \tilde{\Lambda} \end{array}$$

commutes;

- (3)  $\Psi$  and  $\Psi'$  intertwine the covering actions of  $G$  and  $\mathbb{Z}^k[A^{-1}]$ , that is,
  - $\Psi(\alpha(\xi)) = (\tilde{\Phi}^{-1}(\alpha))(\Psi(\xi))$  for all  $\alpha \in \mathbb{Z}^k[A^{-1}]$ ,  $\xi \in \tilde{\mathcal{S}}$ ;
  - $\Psi'(g(x)) = \Psi'(x) + \tilde{\Phi}(g)$  for all  $g \in G$ ,  $x \in \tilde{\Lambda}$ .

Thus the homeomorphism  $\Psi: \tilde{\mathcal{S}} \rightarrow \tilde{\Lambda}$  induces a homeomorphism  $h: \mathcal{S}_A \rightarrow \Lambda$  such that the diagram

$$\begin{array}{ccc} \mathcal{S}_A & \xrightarrow{\sigma_A} & \mathcal{S}_A \\ h \downarrow & & \downarrow h \\ \Lambda & \xrightarrow{f} & \Lambda \end{array}$$

commutes.  $\square$

#### 5.4. Proof of Theorem 1.1.

*Proof of Theorem 1.1.* By Proposition 5.18, Corollary 5.29, and Proposition 5.30 if  $\Lambda$  is not an expanding attractor, then  $\Lambda$  is homeomorphic to a toral solenoid, and  $f|_{\Lambda}$  is conjugate to a solenoidal automorphism.

Furthermore we have  $W^s(x) \cap \Lambda$  is perfect for every  $x \in \Lambda$ . Thus if  $W^s(x) \cap \Lambda$  is locally connected,  $\Lambda$  cannot be expanding, which by the above implies  $\Lambda$  is homeomorphic to a solenoid. However, the only locally connected toral solenoids are in fact tori, that is,  $\mathcal{S}_A$  for  $\det A = \pm 1$ .  $\square$

## 6. PROOF OF COROLLARY 1.2 AND THEOREM 1.3

The following observation is straightforward (see, for example, [4, Lemma 2.4]).

**Lemma 6.1.** *Let  $\Lambda \subset M$  be a compact hyperbolic set for a diffeomorphism  $g: M \rightarrow M$ . The points  $y \in \Lambda$  with the property that  $E^\sigma(y) = \{0\}$  for some  $\sigma \in \{s, u\}$  are periodic and isolated in  $\Lambda$ . In particular if  $\Lambda$  is transitive and contains such a point then  $\Lambda$  is finite.*

We now prove the remaining results from the introduction.

*Proof of Corollary 1.2.* By [16, Theorem 3] and by passing to the inverse if necessary we may assume that  $\Lambda$  is an attractor for  $f$ . We thus have  $\dim E^u|_\Lambda \leq 2$ . Furthermore, by Lemma 6.1,  $\dim E^u|_\Lambda = 0$  would imply that  $\dim(\Lambda) = 0$ , whence we have  $\dim E^u|_\Lambda \geq 1$ . By the spectral decomposition and by passing to an appropriate iterate  $f^n$  we may assume that  $\Lambda$  is topologically mixing for  $f^n$ .

If  $\dim E^u|_\Lambda = 2$  then  $\Lambda$  is a codimension-1 expanding attractor. If  $\dim E^u|_\Lambda = 1$  then by Theorem 1.1 we have that  $\Lambda$  is an embedded toral solenoid. By [11, Theorem 1], no proper 2-dimensional solenoid may be embedded in a closed orientable 3-manifold.

If needed, we first argue on a double cover in the case  $M$  is non-orientable. Also if needed we may pass to a compact manifold with boundary  $N$  containing  $\Lambda$ , and glue a second copy of  $N$  along the boundary to obtain a closed manifold containing  $\Lambda$ . We may then apply [11, Theorem 1] and thus obtain that  $\Lambda$  is locally connected, hence  $\Lambda \cong \mathbb{T}^2$  and  $f^n$  is conjugate to a toral automorphism.  $\square$

*Proof of Theorem 1.3.* We note that for a hyperbolic attractor we always have  $\dim E^u|_\Lambda \leq \dim \Lambda$ .

If  $\dim \Lambda = 0$  we have that  $\dim E^u|_\Lambda = 0$  hence Lemma 6.1 implies that every  $x \in \Lambda$  is periodic and isolated; hence we must have  $\Lambda = \{x\}$  in order for  $\Lambda$  to be topologically mixing. If  $\dim \Lambda = 1$  then Lemma 6.1 implies  $\dim E^u|_\Lambda = 1$ ; hence  $\Lambda$  is an expanding 1-dimensional attractor and  $\Lambda$  is conjugate to the shift map on a generalized 1-solenoid by Theorem II.

If  $\dim \Lambda = 2$ , Lemma 6.1 implies  $1 \leq \dim E^u|_\Lambda \leq 2$ . When  $\dim E^u|_\Lambda = 1$  the fact that  $\Lambda$  is topologically mixing implies  $\Lambda$  is connected, whence  $\Lambda$  is homeomorphic to  $\mathbb{T}^2$  and  $f|_\Lambda$  is conjugate to a hyperbolic toral automorphism by Corollary 1.2. When  $\dim E^u|_\Lambda = 2$  then  $\Lambda$  is a codimension-1 expanding attractor by definition.

Finally, when  $\dim \Lambda = 3$  then Lemma 6.1 implies  $1 \leq \dim E^u|_\Lambda \leq 2$ . Furthermore, [10, Theorem 4.3] implies  $\Lambda$  has non-empty interior, which by [5, Theorem 1] implies  $\Lambda = M$ . But then the result follows from Theorem I.  $\square$

## REFERENCES

- [1] C. Bonatti, Problem in dynamical systems, [http://www.math.sunysb.edu/dynamics/bonatti\\_prob.txt](http://www.math.sunysb.edu/dynamics/bonatti_prob.txt), November 1999.
- [2] H. G. Bothe, *Expanding attractors with stable foliations of class  $C^0$* , in “Ergodic theory and related topics, III,” Lecture Notes in Math., **1514**, Springer, Berlin, 1992, pp. 36–61.
- [3] B. Brenken, *The local product structure of expansive automorphisms of solenoids and their associated  $C^*$ -algebras*, Canad. J. Math., **48** (1996), 692–709.
- [4] A. Brown, *Constraints On Dynamics Preserving Certain Hyperbolic Sets*, Ergodic Theory Dynam. Systems, to appear.
- [5] T. Fisher, *Hyperbolic sets with nonempty interior*, Discrete Contin. Dyn. Syst., **15** (2006), 433–446.



- [6] J. Franks, *Anosov diffeomorphisms*, in “Global Analysis,” Amer. Math. Soc., Providence, R.I., 1970, pp. 61–93.
- [7] V. Z. Grines, V. S. Medvedev, and E. V. Zhuzhoma, *On surface attractors and repellers in 3-manifolds*, Mat. Zametki, **78** (2005), 813–826.
- [8] B. Günther, *Attractors which are homeomorphic to compact abelian groups*, Manuscripta Math., **82** (1994), 31–40.
- [9] K. Hiraide, *A simple proof of the Franks-Newhouse theorem on codimension-one Anosov diffeomorphisms*, Ergodic Theory Dynam. Systems, **21** (2001), 801–806.
- [10] W. Hurewicz and H. Wallman, “Dimension Theory,” Princeton University Press, Princeton, N. J., 1941.
- [11] B. Jiang, S. Wang, and H. Zheng, *No embeddings of solenoids into surfaces*, Proc. Amer. Math. Soc., **136** (2008), 3697–3700.
- [12] J. L. Kaplan, J. Mallet-Paret, and J. A. Yorke, *The Lyapunov dimension of a nowhere differentiable attracting torus*, Ergodic Theory Dynam. Systems, **4** (1984), 261–281.
- [13] A. Katok and B. Hasselblatt, “Introduction to the modern theory of dynamical systems,” Cambridge University Press, Cambridge, 1995.
- [14] A. Manning, *There are no new Anosov diffeomorphisms on tori*, Amer. J. Math., **96** (1974), 422–429.
- [15] S. E. Newhouse, *On codimension one Anosov diffeomorphisms*, Amer. J. Math., **92** (1970), 761–770.
- [16] R. V. Plykin, *The topology of basic sets of Smale diffeomorphisms*, Math. USSR-Sb., **13**(2) (1971), 297–307.
- [17] R. V. Plykin, *Hyperbolic attractors of diffeomorphisms*, Russian Math. Surveys, **35**(3) (1980), 109–121.
- [18] R. V. Plykin, *Hyperbolic attractors of diffeomorphisms (the nonorientable case)*, Russian Math. Surveys, **35**(4) (1980), 186–187.
- [19] D. Ruelle and D. Sullivan, *Currents, flows and diffeomorphisms*, Topology, **14** (1975), 319–327.
- [20] S. Smale, *Differentiable dynamical systems*, Bull. Amer. Math. Soc., **73** (1967), 747–817.
- [21] R. F. Williams, *One-dimensional non-wandering sets*, Topology, **6** (1967), 473–487.
- [22] R. F. Williams, *Classification of one dimensional attractors*, in “Global Analysis,” Amer. Math. Soc., Providence, R.I., 1970, pp. 341–361.
- [23] R. F. Williams, *Expanding attractors*, Inst. Hautes Études Sci. Publ. Math., (1974), 169–203.

DEPARTMENT OF MATHEMATICS, TUFTS UNIVERSITY, MEDFORD, MA 02155

E-mail address: `aaron.brown@tufts.edu`